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Definition and classification of semi-fuzzy quantifiers for the evaluation of fuzzy quantified sentences

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Abstract

This paper describes a classification of semi-fuzzy quantifiers that considerably improves the division between what Zadeh calls quantifiers of the first kind and those of the second kind. A number of cases are contemplated that are not habitually described in the literature on fuzzy quantification (e.g., comparative and exception quantifiers). Models are also defined for all the types of semi-fuzzy quantifiers framed in the classification. Thus in order to construct fuzzy quantifiers it is sufficient to apply a suitable quantifier fuzzification method. This paper also deals with the application of semi-fuzzy quantifiers and fuzzy quantifiers to fuzzy relations. The solution of this problem is of interest in various fields; amongst which, perhaps the most noteworthy is that of fuzzy databases.

Keywords: Fuzzy quantification; Natural language modeling; Quantifier fuzzification mechanisms; Semi-fuzzy quantifiers; Theory of generalized quantifiers; Fuzzy databases

1. Introduction

Fuzzy quantification has been extensively discussed in the literature [2,4–18,23–25,27,29] mainly due to its importance for fields such as fuzzy querying

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in databases, data mining, and the significant increase in expressivity it produces for expressions in fields such as fuzzy expert systems or fuzzy control.

The majority of approaches in the literature use the concept of *fuzzy linguistic quantifier* [28] to represent absolute or proportional fuzzy quantities. Zadeh calls quantifiers used for representing absolute quantities (defined by using fuzzy numbers on $\mathbb{N}$) *quantifiers of the first kind*, and quantifiers used for representing relative quantities (defined by using fuzzy numbers on [0,1]) *quantifiers of the second kind*. In the literature, quantifiers of the first kind are associated to sentences involving only one single fuzzy property (as in "*about three men are tall*" where only "*tall*" is considered to be a fuzzy property); and quantifiers of the second kind are associated to sentences involving two fuzzy properties (as in "*about 70% of blond men are tall*" where "*blond*" and "*tall*" are considered to be fuzzy properties). The linguistic quantifier associated to the former sentence denotes the semantics of "*about 3*" and is defined by using a fuzzy number with domain on $\mathbb{N}$. The linguistic quantifier associated to the second sentence represents the semantics of "*about 70%*" and is defined by using a fuzzy number with domain on [0,1].

This classification is very useful from a practical point of view; nevertheless, it does not contemplate important groups of quantified sentences, such as quantified sentences of exception (e.g., "*all except 3 students are tall*", etc.), or comparative sentences ("*there are about 3 more tall people than blond people*"). There are a number of studies that combine the linguistic and logical perspectives in order to analyze these types of quantified sentences: [3,19–21,26], but it is only recently [14–16,18] that the idea of generalizing models for the evaluation of classic quantified sentences to the fuzzy case has been tackled. In [14] the concept of *semi-fuzzy quantifier* is defined (as being similar to that of fuzzy linguistic quantifier, but more general); and the evaluation of fuzzy quantified sentences by using *quantifier fuzzification mechanisms* of semi-fuzzy quantifiers is presented. This permits the definition of fuzzy quantification models with better behaviour than previous models.

Nevertheless, the description and definition of the principal types of semi-fuzzy quantifiers that can be associated to natural language sentences has not been completely addressed in the literature [14–16,18]. This classification is of undeniable practical interest from the point of view of computing, as when it is combined with the aforementioned fuzzification mechanisms, it supposes an increase in the capacity for handling fuzzy quantified sentences.

This paper presents a description and definition of a broad set of semi-fuzzy quantifiers based on the classic linguistic theory approach of *Generalized Quantifiers* [3,19–21,26]. Firstly, we describe those techniques that make it possible to translate classic linguistic theories to the fuzzy case [14–16,18]. Secondly, we present the classification of semi-fuzzy quantifiers. Finally, we explain how these quantified sentence evaluation techniques can be applied to fuzzy relations. For the sake of clarity, the definition of some quantified sentence evaluation models has been moved to Appendix A.

2. Evaluating quantified sentences

2.1. Quantified sentences in natural language

Quantifiers are principally used in NL as *determiners* (dets) [3,19–21,26] (such as "*the majority*", "*all*", "*these*", etc.) that combine with one or more groups of entities or nouns (Ns) in order to form nominal phrases or determiner phrases (DP).

Fig. 1 shows the syntactic analysis of the sentence "*all men walk*". The determiner "*all*" combines with the noun "*men*" to form the nominal phrase (or determiner phrase) "*all men*". This nominal phrase subsequently combines with the predicate "*walk*" to form the sentence.

From the classic point of view, assertive sentences are either true or false, depending on the model m (or world) in which they are framed. A model consists of a universe E (the set of entities or objects that are discussed in m) and of the denotations assigned to the different elements of the vocabulary being used. Given a model m, predicates are interpreted as subsets (properties or unary relations) in E, and the nominal phrases or determiner phrases are interpreted as *generalized quantifiers (GQs)*, i.e., functions that range from properties to truth values.

For the moment, we will assume that the denotation of both names and predicates is given by unary properties. In Section 4 we will explain how to

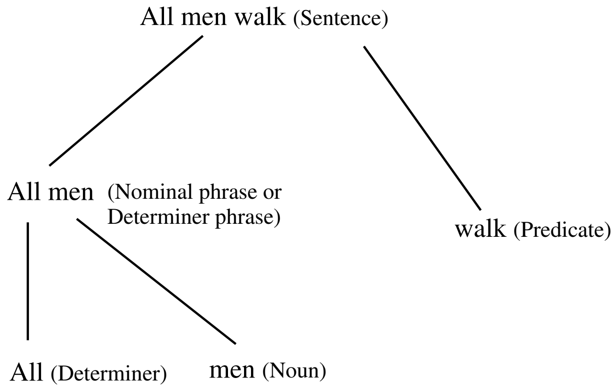

Fig. 1. Syntactic analysis of the quantified sentence "all men walk".

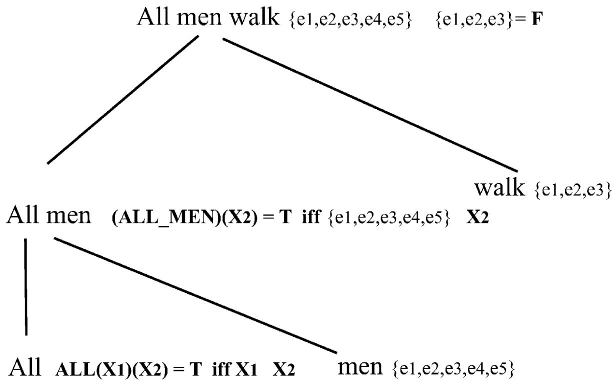

Fig. 2. Semantic analysis of the sentence "all men walk" for the referential set $E = \{e_1, \ldots, e_8\}$ and denotations *men* $= \{e_1, \ldots, e_5\}$ and *walk* $= \{e_1, \ldots, e_3\}$.

combine generalized quantifiers with properties whose denotation is given by relations of arity greater than 1.

Returning to the sentence in the previous example, the denotation of "*walk*" is the subset of the elements of the referential universe E which have the property of walking; the denotation of "*men*" is the subset of elements of E that have the property of being men; and the denotation of "*all*" is a function [1] $\mathbf{all}_E : (X_1 \in \wp(E)) \to (f : X_2 \in \wp(E) \to \{0,1\})$ defined as

$$\mathbf{all}_E(X_1)(X_2) = 1 \quad \text{iff } X_1 \subseteq X_2 \tag{1}$$

where $\wp(E)$ denotes the powerset of E.

Fig. 2 shows the denotations of the lexical elements in the sentence "*all men walk*" for a particular situation and its evaluation.

In the last example, the determiner "*all*" combines with a single noun "*men*" in order to form the determiner phrase. This kind of determiners are sometimes called in the literature Det_1. In [20] determiners which combine with more than one nominal phrase, as in the example "*more students than teachers come to the party*", are considered. In this case, the denotation of determiner "*more ... than*" is defined as a function:

$$(\mathbf{more \ldots than})_E(X_1, X_2)(X_3) = 1 \quad \text{iff } |X_1 \cap X_3| > |X_2 \cap X_3| \tag{2}$$

In this example, the determiner "*more ... than*" is said to be a Det_2 determiner, since it combines with two nouns in order to form the determiner phrase.

[1] In linguistics, generalized quantifiers are sometimes viewed as relations. For example, the denotation of "*all*" can be interpreted as a relation $R_{\mathrm{all}} \subset \wp(E)^2 / (X_1, X_2) \in R_{\mathrm{all}}$ iff $X_1 \subseteq X_2$.

It should be stressed that in expressions (1) and (2) we are distinguishing between the groups of properties that are combined with the semantics of the determiner (in the above with the properties X_1 and X_2, which denote noun semantics) and the property which combines with the semantics of the determiner phrase (in this case with the property X_3 which denotes the semantics of the verbal phrase).

In order to express the role of nominal and verbal phrases in the denotation of determiners, we introduce the notation that is used in [20], where the following classes of functions are defined:

$$\begin{aligned} \mathit{TYPE}\langle 1\rangle &= [\wp(E) \to \mathbf{2}] \\ \mathit{TYPE}\langle 1,1\rangle &= [\wp(E) \to \mathit{TYPE}\langle 1\rangle] \\ \mathit{TYPE}\langle\langle 1,1\rangle,1\rangle &= [\wp(E)^2 \to \mathit{TYPE}\langle 1\rangle] \end{aligned}$$

where $[X \to Y]$ denotes the set of functions with domain X and range Y.

Thus,

$$\begin{aligned} \mathbf{all} &\in \mathit{TYPE}\langle 1,1\rangle \\ \mathbf{more}\ldots\mathbf{than} &\in \mathit{TYPE}\langle\langle 1,1\rangle,1\rangle \end{aligned}$$

In this notation, the first element of the pair $\langle a,b\rangle$ represents the number and the arity of the nominal phrases that are combined with the determiner; the second element of the pair represents the number and the arity of the predicates.

The notation used to represent the denotation of determiners (i.e., the one used in the expression (1)) is suitable from a linguistic point of view, since once the assignation of meaning to the terms in the lexis has been performed, evaluation of quantified sentences can be carried out in a compositional manner (from bottom to top of the syntactic tree). Nevertheless, the following notation [14,15] is easier to handle from a mathematical perspective:

Definition 1 (*Classic quantifier* [14]). A classic s-ary quantifier on a referential set E is a function $Q : \wp(E)^s \to \{0,1\}$.

Examples of some definitions of classic quantifiers are:

$$\begin{aligned} \mathbf{all}(X_1,X_2) &= \begin{cases} 0 & X_1 \nsubseteq X_2 \\ 1 & X_1 \subseteq X_2 \end{cases} \\ \mathbf{at\_least}\ 80\%\ \mathbf{of\_the}(X_1,X_2) &= \begin{cases} \dfrac{|X_1 \cap X_2|}{|X_1|} \geqslant 0.8 & X_1 \neq \emptyset \\ 1 & X_1 = \emptyset \end{cases} \end{aligned} \tag{3}$$

The latter notation [2] is generally used throughout the paper.

2.2. Evaluation of quantified sentences in the fuzzy case

In [14] a working scheme is formulated which extends the *Theory of Generalized Quantifiers* [3,19–21,26] to the fuzzy case. In this section we will expound this scheme.

Here we will employ the notation $\mathbf{I} = [0,1]$, and $\widetilde{\wp}(E)$ for the fuzzy powerset of E. The fuzzy analogue of classic quantifiers is defined as follows:

Definition 2 (*Fuzzy quantifier* [14,16]). An s-ary fuzzy quantifier $\widetilde{Q}$ on a base set $E \neq \emptyset$ is a mapping $\widetilde{Q} : \widetilde{\wp}(E)^s \to \mathbf{I}$ which to each choice of $X_1, \ldots, X_s \in \widetilde{\wp}(E)$ assigns a gradual result $\widetilde{Q}(X_1, \ldots, X_s) \in \mathbf{I}$.

An example of a fuzzy quantifier is $\widetilde{\mathbf{all}} : \widetilde{\wp}(E)^2 \to \mathbf{I}$, which in principle may be defined as

$$\widetilde{\mathbf{all}}(X_1, X_2) = \inf\{\max(1 - \mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\} \tag{4}$$

Example 1. Let us consider the evaluation of the sentence "*all tall people are blond*" in a referential set $E = \{e_1, \ldots, e_4\}$. Let us assume the predicates "*tall*" and "*blond*" are given by the following subsets

$$X_1 = \{0.8/e_1, 1/e_2, 0.6/e_3, 0.3/e_4\}, \quad X_2 = \{0.9/e_1, 0.7/e_2, 0.3/e_3, 0.2/e_4\}$$

Using (4) we have

$$\begin{aligned} \widetilde{\mathbf{all}}(X_1, X_2) &= \inf\{\max(1 - \mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\} \\ &= \inf\{0.9, 0.7, 0.4, 0.7\} = 0.4 \end{aligned}$$

In the same way as for classic quantifiers, there is no problem in defining a function $\widetilde{\mathbf{all}}'$ such that $\widetilde{\mathbf{all}}'(X_1)(X_2) = \widetilde{\mathbf{all}}(X_1, X_2)$.

Although a certain consensus may be achieved for accepting (4) as a suitable definition for the fuzzy quantifier $\widetilde{\mathbf{all}}$, this is not the usual case. Very often, it is extremely difficult to define a suitable expression for outlining the evaluation of a quantified sentence. In [14] this problem is avoided by introducing the concept of semi-fuzzy quantifier, which is a half-way point between classic quan-

[2] These two notations are equivalent, since we can define the functions $\mathbf{all}'_E : \wp(E) \to (\wp(E) \to \mathbf{I})$, $\mathbf{more}\ldots\mathbf{than}'_E : \wp(E)^2 \to (\wp(E) \to \mathbf{I})$ from the functions $\mathbf{all}_E : \wp(E)^2 \to \mathbf{I}$, $\mathbf{more}\ldots\mathbf{than}_E : \wp(E)^3 \to \mathbf{I}$:

$$\mathbf{all}'_E(X_1)(X_2) = \mathbf{all}_E(X_1, X_2)$$

$$\mathbf{more}\ldots\mathbf{than}'_E(X_1, X_2)(X_3) = \mathbf{more}\ldots\mathbf{than}_E(X_1, X_2, X_3)$$

tifiers and fuzzy quantifiers, and is very close to the idea of Zadeh's linguistic quantifier [29]. Semi-fuzzy quantifiers are similar to classic quantifiers, nevertheless they allow variation of the results in the [0,1] interval.

Definition 3 (*Semi-fuzzy quantifier* [14,16]). An *s-ary* semi-fuzzy quantifier Q on a base set $E \neq \emptyset$ is a mapping $Q : \wp(E)^s \to \mathbf{I}$ which to each choice of crisp $X_1, \ldots, X_s \in \wp(E)$ assigns a gradual result $Q(X_1, \ldots, X_s) \in \mathbf{I}$.

Examples of semi-fuzzy quantifiers are:

$$\textbf{about\_5}\ (X_1, X_2) = T_{2,4,6,8}(|X_1 \cap X_2|)$$

$$\textbf{about\_80\%\_or\_more\_of\_the}\ (X_1, X_2) = \begin{cases} S_{0.5,0.8}\left(\dfrac{|X_1 \cap X_2|}{|X_1|}\right) & X_1 \neq \emptyset \\ 1 & X_1 = \emptyset \end{cases}$$

where $T_{a,b,c,d}(x)$ represents a trapezoidal function with kernel $[b, c]$ and support (a, d), and $S_{\alpha,\gamma}(x)$ is the Zadeh's S-function. Both are represented in Fig. 3. [3]

Example 2. Let us consider the evaluation of the sentence "*about* 80% *or more of the cars have electric windows*" on a referential set $E = \{e_1, \ldots, e_6\}$ where predicates "*car*" and "*having electric windows*" are represented by:

$$X_1 = \{e_1, \ldots, e_5\}, \quad X_2 = \{e_1, e_2, e_3, e_6\}$$

[3] Functions $T_{a,b,c,d}$ and $S_{\alpha,\gamma}$ are defined as

$$T_{a,b,c,d}(x) = \begin{cases} 0 & x \leqslant a \\ \dfrac{x-a}{b-a} & a < x \leqslant b \\ 1 & b < x \leqslant c \\ 1 - \dfrac{x-c}{d-c} & c < x \leqslant d \\ 0 & d < x \end{cases}, \quad S_{\alpha,\gamma}(x) = \begin{cases} 0 & x < \alpha \\ 2\left(\dfrac{(x-\alpha)}{(\gamma-\alpha)}\right)^2 & \alpha < x \leqslant \dfrac{\alpha+\gamma}{2} \\ 1 - 2\left(\dfrac{(x-\gamma)}{(\gamma-\alpha)}\right)^2 & \dfrac{\alpha+\gamma}{2} < x \leqslant \gamma \\ 1 & \gamma < x \end{cases}$$

In this paper we do not assume convexity for fuzzy numbers, in order to allow the treatment of non-convex semi-fuzzy quantifiers such as

$$Q(X) = \begin{cases} 0 & |X| \text{ is odd} \\ 1 & |X| \text{ is even} \end{cases}$$

which would be associated to the determiner "*an even number of*". Furthermore, we occasionally take $\mathbb{R} \cup \{\infty\}$ as the definition universe of fuzzy numbers.

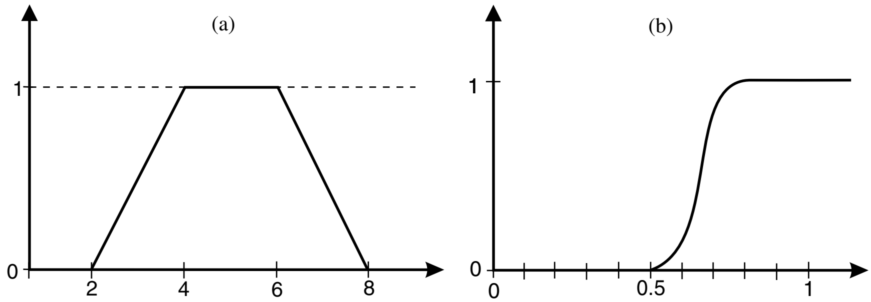

Fig. 3. Fuzzy numbers for the definition of semi-fuzzy quantifiers **about_5** (a) and **about_80%_or_more** (b).

then

$$\mathbf{about\_80\%\_or\_more}_E(X_1, X_2) = S_{0.5,0.8}\left(\frac{|X_1 \cap X_2|}{|X_1|}\right) = 0.22$$

Semi-fuzzy quantifiers are much more intuitive and easier to define than fuzzy quantifiers, but they do not resolve the problem of evaluating fuzzy quantified sentences. In order to do so mechanisms are needed [16] that enable us to transform semi-fuzzy quantifiers into fuzzy quantifiers, i.e., mappings with domain in the universe of semi-fuzzy quantifiers and range in the universe of fuzzy quantifiers:

$$F : (Q : \wp(E)^s \mapsto \mathbf{I}) \mapsto (\widetilde{Q} : \widetilde{\wp}(E)^s \mapsto \mathbf{I}) \tag{5}$$

Following the terminology employed in [16], these mechanisms are called *quantifier fuzzification mechanisms*.

In Appendix A two mechanisms for extending semi-fuzzy quantifiers to fuzzy quantifiers are defined. In [14,16,18] a series of axioms that guarantee a suitable behaviour for a quantifier fuzzification mechanism are defined. Quantifier fuzzification mechanism that satisfy these axioms are called *Determiner Fuzzification Schemes* (DFS). In [8,10–12] a number of probabilistic mechanisms are defined. [4]

Thus the existence of suitable mechanisms for the fuzzification of semi-fuzzy quantifiers makes it possible to translate the problem of defining fuzzy quantifiers to that of defining semi-fuzzy quantifiers. In the following section we present the definition and classification of a wide set of semi-fuzzy quantifiers.

[4] The mechanisms defined in these works are not quantifier fuzzification mechanisms in the sense of expression (5), since their applicability is only guaranteed in finite domains. Nevertheless, this is sufficient for practical applications, and the behaviour of the models is adequate.

In order to obtain the corresponding fuzzy models an appropriate quantifier fuzzification mechanism needs to be applied to these semi-fuzzy quantifiers.

3. Classification

The aim of this section is to present a classification of semi-fuzzy quantifiers that, once transformed into fuzzy quantifiers by means of a quantifier fuzzification mechanism, may be suitable for the evaluation of a wide group of quantified sentences. Furthermore, whenever possible, we attempt to relate the semi-fuzzy quantifiers with the determiners of the corresponding denotation.

In order to evaluate a given sentence it will generally be possible to use different semi-fuzzy quantifiers. In the explanation we comment on various transformations that make it possible to transform semi-fuzzy quantifiers of a specific type into others of a different type. Fulfillment of these transformations (that are valid for semi-fuzzy quantifiers) also for fuzzy quantifiers depends on the characteristics of the quantifier fuzzification mechanism that is being used. Throughout the presentation we will briefly point out the validity of the transformations mentioned for the quantifier fuzzification mechanisms explained in Appendix A, although we do not overly emphasize these aspects as they are beyond the objectives of this paper. For further information the reader should consult references [11,14,16,18].

We will consider those quantified sentences that are associated with quantitative determiners, i.e., those which are not affected by permutations in the argument sets [16,18,20]. These include the most important types from a practical point of view. In Section 5 we comment on this and other cases that are not dealt with here. We also suppose that the referential E is finite, which is sufficient from an applicational perspective.

We use the following notation

$$Q^{\mathrm{FN}}_{E,a,\mathrm{type}} \quad \text{or} \quad Q^{\mathrm{FN},v}_{E,a,\mathrm{type}} \tag{6}$$

to indicate, in a parametrized manner, how semi-fuzzy quantifiers can be defined. Here, FN is a fuzzy number on $\mathbb{Z}$, [0,1] or $\mathbb{R}$; $v \in [0,1]$; E is the universe of reference; $a \in \mathbb{N}$ indicates the arity of the semi-fuzzy quantifier; and *type* is a parameter that enables us to distinguish between different options for constructing semi-fuzzy quantifiers. This notation will also be useful for explaining the relations between different types of semi-fuzzy quantifiers.

3.1. Unary semi-fuzzy quantifiers

By using unary semi-fuzzy quantifiers evaluation of sentences involving a single property (e.g., "*all people are tall*", "*about* 40% *of people are tall*", "*more than* 3 *people are tall*") can be performed. As we will see later on, it is generally more natural to associate binary semi-fuzzy quantifiers to these types of sentences.

Definition 4 (*Unary cardinal semi-fuzzy quantifier*). A semi-fuzzy quantifier $Q : \wp(E) \to \mathbf{I}$ is an unary cardinal semi-fuzzy quantifier if there exists a fuzzy number $\mathrm{FN} : \mathbb{N} \to \mathbf{I}$ such that

$$Q(X) = \mathrm{FN}(|X|)$$

We will consider two different manners of constructing unary cardinal semi-fuzzy quantifiers (additional possibilities are given in Section 3.5). The first one, by means of an absolute fuzzy number $\mathrm{FN} : \mathbb{N} \to \mathbf{I}$

$$Q^{\mathrm{FN}}_{E,1,\mathrm{card}}(X) = \mathrm{FN}(|X|)$$

(where the notation (6) is being employed). The second one, by means of a proportional fuzzy number $\mathrm{FN} : [0,1] \to \mathbf{I}$

$$Q^{\mathrm{FN}}_{E,1,\mathrm{prop}}(X) = \mathrm{FN}\left(\frac{|X|}{|E|}\right)$$

The following examples show how the definitions given above can be used in the evaluation of quantified sentences.

Example 3. In order to evaluate the sentence "about 5 people are students" the following semi-fuzzy quantifier can be defined:

$$Q^{T_{1,4,6,9}}_{E,1,\mathrm{card}}(X) = T_{1,4,6,9}(|X|)$$

where $T_{1,4,6,9}(x)$ is a trapezoidal function with kernel [4,6] and support (1,9), and E the referential of individuals.

Let us denote by Y the set of people who are students:

$$Y = \{1/e_1, 1/e_2, 1/e_3, 0/e_4, \ldots, 0/e_{10}\}$$

Then

$$Q^{T_{1,4,6,9}}_{E,1,\mathrm{card}}(Y) = T_{1,4,6,9}(|Y|) = \tfrac{2}{3}$$

In the fuzzy case we need to use a quantifier fuzzification mechanism (see Appendix A). Let us consider the evaluation of the sentence "around 5 people are blond", where the set denoting the blond people is

$$Y = \{1/e_1, 0.8/e_2, 1/e_3, 0.6/e_4, 0/e_5, \ldots, 0/e_{10}\}$$

Table 1 shows the use of the mechanism F^I (see Appendix A), and the result is

$$F^I(Q^{T_{1,4,6,9}}_{E,1,\mathrm{card}})(Y) = \int_0^1 Q^{T_{1,4,6,9}}_{E,1,\mathrm{card}}(Y_\alpha)\,\mathrm{d}\alpha = 0.85$$

Table 1
Application of the mechanism F^I

	$Y_{\geqslant \alpha}$	$Q_{E,1,\mathrm{card}}^{T_{1,4,6,9}}(Y_{\geqslant \alpha})$
$\alpha \in (0.8, 1]$	$\{e_1, e_3\}$	0.5
$\alpha \in (0.6, 0.8]$	$\{e_1, e_2, e_3\}$	0.75
$\alpha \in (0, 0.6]$	$\{e_1, e_2, e_3, e_4\}$	1

Example 4. In order to evaluate the sentence "*around* 70% *or more of the people are students*" the following semi-fuzzy quantifier can be defined

$$Q_{E,1,\mathrm{prop}}^{S_{0.5,0.7}}(X) = S_{0.5,0.7}\left(\frac{|X|}{|E|}\right)$$

When this semi-fuzzy quantifier is applied over the set of people that are students, defined as in the previous example, we obtain:

$$Q_{E,1,\mathrm{prop}}^{S_{0.5,0.7}}(Y) = Q_{E,1,\mathrm{prop}}^{S_{0.5,0.7}}(\{1/e_1, \ldots, 1/e_8, 0/e_{10}\}) = 1$$

As in the previous example, in the fuzzy case a quantifier fuzzification mechanism has to be used. Let Y be the fuzzy set denoting the people that are blond, as in the previous example. Table 2 shows the use of the mechanism M (see Appendix A), with which we obtain

$$M(Q_{E,1,\mathrm{prop}}^{S_{0.5,0.7}})(Y) = \int_0^1 (Q_{E,1,\mathrm{prop}}^{S_{0.5,0.7}})_\gamma(Y)\,\mathrm{d}\gamma = 0.6$$

It is important to point out that the expressions given above enable us to express the classic universal and existential quantifiers:

$$\forall_E(X) = Q_{E,1,\mathrm{prop}}^{\mathrm{FN_{all}}}(X)$$
$$\exists_E(X) = Q_{E,1,\mathrm{card}}^{\mathrm{FN_{\exists}}}(X)$$

where $\mathrm{FN_{all}}$ and $\mathrm{FN_{\exists}}$ are defined as

Table 2
Application of the mechanism M

	$(Y)_\gamma^{\min}$	$(Y)_\gamma^{\max}$	$(Q_{E,1,\mathrm{prop}}^{S_{0.5,0.7}})_\gamma(Y)$
$\gamma \in [0, 0.2]$	$\{e_1, e_2, e_3, e_4\}$	$\{e_1, e_2, e_3, e_4\}$	1
$\gamma \in (0.2, 0.6]$	$\{e_1, e_2, e_3\}$	$\{e_1, e_2, e_3, e_4\}$	0.5
$\gamma \in (0.6, 1]$	$\{e_1, e_3\}$	$\{e_1, e_2, e_3, e_4\}$	0.5

$$\mathrm{FN}_{\forall}(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}, \quad x \in [0,1]$$

$$\mathrm{FN}_{\exists}(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}, \quad x \in \mathbb{N}$$

Note that for finite E the following transformation can be used:

$$Q^{\mathrm{FN}}_{E,1,\mathrm{prop}}(X) = \mathrm{FN}\left(\frac{X}{|E|}\right) = (\mathrm{FN} \otimes |E|)(X) = Q^{\mathrm{FN}\otimes|E|}_{E,1,\mathrm{card}}(X), \quad E \neq \emptyset \tag{7}$$

where $\otimes$ represents the product between fuzzy numbers.

Transformation (7) makes evident that for a finite referential E expressions $Q^{\mathrm{FN}}_{E,1,\mathrm{card}}$ and $Q^{\mathrm{FN}}_{E,1,\mathrm{prop}}$ define the same set of semi-fuzzy quantifiers. Nevertheless it is useful, from the applications point of view, to keep both types of expressions.

3.2. *Binary semi-fuzzy quantifiers*

3.2.1. *Binary absolute semi-fuzzy quantifiers*

This section deals with binary absolute semi-fuzzy quantifiers, which are closely related to determiners of cardinal and co-cardinal denotation Det_1 [19].

First, we consider the group of semi-fuzzy quantifiers that are associated to determiners of cardinal denotation, which will enable us to formulate the evaluation of sentences of the type: "*about* 6 *tall people are blond*", "*some tall people are blond*", etc.

Definition 5 (*Binary cardinal semi-fuzzy quantifier*). A semi-fuzzy quantifier $Q : \wp(E)^2 \to \mathbf{I}$ is a binary cardinal semi-fuzzy quantifier if there exists a fuzzy number $\mathrm{FN} : \mathbb{N} \to \mathbf{I}$ such that

$$Q(X_1, X_2) = \mathrm{FN}(|X_1 \cap X_2|)$$

In [18] these quantifiers are named absolute. In this paper we include co-cardinal quantifiers (explained below) in the group of absolute quantifiers.

Binary cardinal semi-fuzzy quantifiers depend on $|X_1 \cap X_2|$; i.e., given $X_1, X_2, X_1', X_2' \in \wp(E)$ such that $|X_1 \cap X_2| = |X_1' \cap X_2'|$ then it holds that $Q(X_1, X_2) = Q(X_1', X_2')$.

Let $\mathrm{FN} : \mathbb{N} \to \mathbf{I}$. Using notation (6) we define

$$Q^{\mathrm{FN}}_{E,2,\mathrm{card}}(X_1, X_2) = \mathrm{FN}(|X_1 \cap X_2|)$$

Example 5. In order to evaluate the sentence "*about* 5 *students are Spanish*" we can formulate the semi-fuzzy quantifier

$$Q^{\mathrm{FN}}_{E,2,\mathrm{card}}(X_1,X_2) = T_{1,4,6,9}(|X_1 \cap X_2|)$$

When this semi-fuzzy quantifier is applied to the sets X_1, X_2, that respectively represent those individuals that are students and those that are Spanish

$$X_1 = \{1/e_1,\ldots,1/e_8,0/e_9,0/e_{10}\}$$
$$X_2 = \{1/e_1,1/e_2,1/e_3,0/e_4,\ldots,0/e_{10}\}$$

we obtain

$$Q^{T_{1,4,6,9}}_{E,2,\mathrm{card}}(X_1,X_2) = T_{1,4,6,9}(|X_1 \cap X_2|) = \tfrac{2}{3}$$

These semi-fuzzy quantifiers can be defined in terms of unary semi-fuzzy quantifiers, since [5]

$$Q^{\mathrm{FN}}_{E,2,\mathrm{card}}(X_1,X_2) = \mathrm{FN}(|X_1 \cap X_2|) = Q^{\mathrm{FN}}_{E,1,\mathrm{card}}(X_1 \cap X_2)$$

We now go on to deal with the group of semi-fuzzy quantifiers that are associated to co-cardinal denotation determiners, which will enable us to formulate the evaluation of sentences like: "*all tall people are blond*", "*all except* 3 *tall people are blond*", etc. In order to do so, we introduce the following expression:

Definition 6 (*Binary co-cardinal semi-fuzzy quantifiers*). A semi-fuzzy quantifier $Q : \wp(E)^2 \to \mathbf{I}$ is a binary co-cardinal semi-fuzzy quantifier if there exists a fuzzy number $\mathrm{FN} : \mathbb{N} \to \mathbf{I}$ such that

$$Q(X_1,X_2) = \mathrm{FN}(|X_1 \setminus X_2|)$$

In [18] these quantifiers are named quantifiers of exception.

Binary co-cardinal semi-fuzzy quantifiers only depend on $|X_1 \setminus X_2|$; i.e., given $X_1,X_2,X_1',X_2' \in \wp(E)$ such that $|X_1 \setminus X_2| = |X_1' \setminus X_2'|$ then it holds that $Q(X_1,X_2) = Q(X_1',X_2')$.

Let $\mathrm{FN} : \mathbb{N} \to \mathbf{I}$. Using the notation (6) we define

$$Q^{\mathrm{FN}}_{E,2,\mathrm{co\_card}}(X_1,X_2) = \mathrm{FN}(|X_1 \setminus X_2|)$$

Binary cardinal and co-cardinal semi-fuzzy quantifiers are related to linguistic quantifiers of the first kind, according to the definition given by Zadeh [29].

[5] Although the aforementioned transformation is valid for the case of semi-fuzzy quantifiers, its validity for fuzzy quantifiers depends on the fuzzy quantifier fuzzification mechanism that is used. The transformation is valid when we use the mechanism $\mathcal{M}$ (see Appendix A) but not for the mechanism $\mathcal{F}^I$ (a counter-example for this case is $X_1 = X_2 = \{0.5/e_1\}$ and $Q^{\mathrm{FN}_\exists}_{E,2,\mathrm{card}}$).

Example 6. In order to evaluate the sentence "*all except about five students are Spanish*" the following semi-fuzzy quantifier can be defined:

$$Q^{T_{1,4,6,9}}_{E,2,\mathrm{co\_card}}(X_1, X_2) = T_{1,4,6,9}(|X_1 \setminus X_2|)$$

When this semi-fuzzy quantifier is applied to the same sets X_1, X_2 defined in the previous example we obtain

$$Q^{T_{1,4,6,9}}_{E,2,\mathrm{co\_card}}(X_1, X_2) = T_{1,4,6,9}(|X_1 \setminus X_2|) = 0$$

The following relation holds between cardinal and co-cardinal semi-quantifiers: [6]

$$\begin{aligned} Q^{\mathrm{FN}}_{E,2,\mathrm{co\_card}}(X_1, X_2) &= \mathrm{FN}(|X_1 \setminus X_2|) \\ &= \mathrm{FN}(|X_1 \cap \neg X_2|) \\ &= Q^{\mathrm{FN}}_{E,2,\mathrm{card}}(X_1, \neg X_2) \end{aligned}$$

3.2.2. Binary proportional semi-fuzzy quantifiers

This section deals with proportional semi-fuzzy quantifiers, which are related to determiners of proportional denotation (*Det*$_2$) [20]. Examples of this type of determiners are "the majority", "about 70% or more", "all except about a tenth", etc.

Definition 7 (*Binary proportional semi-fuzzy quantifier*). A semi-fuzzy quantifier $Q : \wp(E)^2 \to \mathbf{I}$ is a binary proportional semi-fuzzy quantifier if there exists a proportional fuzzy number $\mathrm{FN} : [0,1] \to \mathbf{I}$ such that

$$Q(X_1, X_2) = \begin{cases} \mathrm{FN}\left(\frac{|X_1 \cap X_2|}{|X_1|}\right) & X_1 \neq \emptyset \\ v \in [0,1] & X_1 = \emptyset \end{cases} \tag{8}$$

The definition above is similar to that presented in [15].

If similar semantics are assumed for determiners "all" and "100%" value $v = 1$ should be assigned to the indetermination situation $X_1 = \emptyset$.

Proportional semi-fuzzy quantifiers correspond with quantifiers of the second kind (or relative quantifiers), according to the definition given by Zadeh [28].

Let $\mathrm{FN} : [0,1] \to \mathbf{I}$ a fuzzy number, $v \in [0,1]$. Using the notation (6) we define

[6] Both quantifier fuzzification mechanisms described in Appendix A fulfill this transformation.

$$Q_{E,2,\mathrm{prop}}^{\mathrm{FN},v}(X_1,X_2) = \begin{cases} \mathrm{FN}\left(\dfrac{|X_1 \cap X_2|}{|X_1|}\right) & X_1 \neq \emptyset \\ v & X_1 = \emptyset \end{cases}$$

Example 7. In order to evaluate the sentence "*about* 60% *of the students are Spanish*" the following semi-fuzzy quantifier can be defined:

$$Q_{E,2,\mathrm{prop}}^{S_{0.5,0.6},1}(X_1,X_2) = \begin{cases} S_{0.5,0.6}\left(\dfrac{|X_1 \cap X_2|}{|X_1|}\right) & X_1 \neq \emptyset \\ 1 & X_1 = \emptyset \end{cases}$$

When this semi-fuzzy quantifier is applied to the same sets X_1, X_2 as in the previous example we obtain

$$\begin{aligned} Q_{E,2,\mathrm{prop}}^{S_{0.5,0.6},1}(X_1,X_2) &= Q_{E,2,\mathrm{prop}}^{S_{0.5,0.6},1}(X_1,X_2) \\ &= S_{0.5,0.6}\left(\frac{|X_1 \cap X_2|}{|X_1|}\right) \\ &= S_{0.5,0.6}\left(\frac{3}{8}\right) = 0 \end{aligned}$$

In the same manner as for binary absolute semi-fuzzy quantifiers, it is possible to define a class of *co-proportional* semi-fuzzy quantifiers (e.g., "all except 10%", etc.), although this is not strictly necessary, since one group can be expressed according to the other.

It should also be stated that both binary absolute and binary proportional semi-fuzzy quantifiers are naturally associated to determiners of denotation $TYPE\langle 1,1\rangle$.

3.2.3. Binary comparative semi-fuzzy quantifiers

Binary comparative semi-fuzzy quantifiers cannot be associated to natural language sentences in such an inmediate manner as the other binary semi-fuzzy quantifiers. They can be associated with some existential sentences [19] such as "there are more men than women", "there are about three more women than men", "there are twice as many men as women", etc. In any case, and similarly to the case of unary semi-fuzzy quantifiers, it may be more natural to associate ternary semi-fuzzy quantifiers to these types of sentences.

Definition 8 (*Binary cardinal comparative semi-fuzzy quantifier*). A semi-fuzzy quantifier $Q : \wp(E)^2 \to \mathbf{I}$ is a binary cardinal comparative semi-fuzzy quantifier if there exists a fuzzy number $\mathrm{FN} : \mathbb{Z} \to \mathbf{I}$ such that

$$Q(X_1,X_2) = \mathrm{FN}(|X_1| - |X_2|)$$

Let FN : $\mathbb{Z} \to \mathbf{I}$ a fuzzy number. We define

$$Q^{\mathrm{FN}}_{E,2,\mathrm{card\_comp}}(X_1, X_2) = \mathrm{FN}(|X_1| - |X_2|)$$

Example 8. In order to evaluate the sentence "*there are about five more men than women*" the following semi-fuzzy quantifier can be defined

$$Q^{T_{1,4,6,9}}_{E,2,\mathrm{card\_comp}}(X_1, X_2) = T_{1,4,6,9}(|X_1| - |X_2|)$$

When we apply this semi-fuzzy quantifier onto the sets X_1, X_2, which represent those individuals that are men and those that are women, respectively,

$$X_1 = \{1/e_1, \ldots, 1/e_8, 0/e_9, 0/e_{10}\}$$
$$X_2 = \{1/e_1, 1/e_2, 1/e_3, 0/e_4, \ldots, 0/e_{10}\}$$

the result is

$$\begin{aligned} Q^{T_{1,4,6,9}}_{E,2,\mathrm{card\_comp}}(X_1, X_2) &= Q^{T_{1,4,6,9}}_{E,2,\mathrm{card\_comp}}(X_1, X_2) \\ &= T_{1,4,6,9}(|X_1| - |X_2|) = 1 \end{aligned}$$

Definition 9 (*Binary proportional comparative semi-fuzzy quantifier*). A semi-fuzzy quantifier $Q : \wp(E)^2 \to \mathbf{I}$ is a binary proportional comparative semi-fuzzy quantifier if there exists a fuzzy number FN : $\mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ such that

$$Q(X_1, X_2) = \begin{cases} \mathrm{FN}\left(\frac{|X_1|}{|X_2|}\right) & X_2 \neq \emptyset \\ \mathrm{FN}(\infty) & X_1 \neq \emptyset \wedge X_2 = \emptyset \\ \mathrm{FN}(1) & X_1 = \emptyset \wedge X_2 = \emptyset \end{cases}$$

Let FN : $\mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ be a fuzzy number. We define

$$Q^{\mathrm{FN}}_{E,2,\mathrm{prop\_comp}}(X_1, X_2) = \begin{cases} \mathrm{FN}\left(\frac{|X_1|}{|X_2|}\right) & X_2 \neq \emptyset \\ \mathrm{FN}(\infty) & X_1 \neq \emptyset \wedge X_2 = \emptyset \\ \mathrm{FN}(1) & X_1 = \emptyset \wedge X_2 = \emptyset \end{cases}$$

Example 9. In order to evaluate the sentence"*there are twice as many men as women*" we can formulate the semi-fuzzy quantifier

$$Q^{T_{1,2,2,3}}_{E,2,\mathrm{prop\_comp}}(X_1, X_2) = \begin{cases} T_{1,2,2,3}\left(\frac{|X_1|}{|X_2|}\right) & X_2 \neq \emptyset \\ 0 & X_1 \neq \emptyset \wedge X_2 = \emptyset \\ 0 & X_1 = \emptyset \wedge X_2 = \emptyset \end{cases}$$

It should be noted that these semi-fuzzy quantifiers do not depend on the intersection of the properties, whilst the remaining binary semi-fuzzy quantifiers (absolute and proportional) do depend, in some manner, on the intersection of the properties.

3.3. Ternary semi-fuzzy quantifiers

This section deals with ternary semi-fuzzy quantifiers. These semi-fuzzy quanfiers are related to sentences involving two nominal phrases and one predicate, or one nominal phrase and two predicates.

The following are examples of sentences that can be evaluated by means of ternary semi-fuzzy quantifiers:

"More students than teachers come to the party".
"Almost as many students as teachers come to the party".
"There are about five more men than women".
"More than 10 times as many students as teachers come to the party".
"About twice or more students are at the lake than students are at the party".

In the first sentence, the determiner *more ... than* (Det_2) has the following denotation:

$$(\mathbf{more}\ X_1\ \mathbf{than}\ X_2)_E(X_3) = \mathit{True} \quad \text{iff } |X_1 \cap X_3| > |X_2 \cap X_3|$$

In [21] the determiners associated to the semi-fuzzy quantifiers defined in this section are classified as cardinal comparatives.

In order to characterize the semi-fuzzy quantifiers associated to this type of determiners we introduce the following definitions:

Definition 10 (*Ternary cardinal comparative*, *type* $\langle\langle 1,1\rangle,1\rangle$). A semi-fuzzy quantifier $Q : \wp(E)^3 \to \mathbf{I}$ is a ternary cardinal comparative semi-fuzzy quantifier of type $\langle\langle 1,1\rangle,1\rangle$ if there exists a fuzzy number $\mathrm{FN} : \mathbb{Z} \to \mathbf{I}$ such that

$$Q_E(X_1, X_2, X_3) = \mathrm{FN}(|X_1 \cap X_3| - |X_2 \cap X_3|)$$

Let $\mathrm{FN} : \mathbb{Z} \cup \{\infty\} \to \mathbf{I}$ be a fuzzy number. We define

$$Q^{\mathrm{FN}}_{E,3,\mathrm{card\_comp\_}\langle\langle 1,1\rangle,1\rangle} = \mathrm{FN}(|X_1 \cap X_3| - |X_2 \cap X_3|)$$

Example 10. In order to evaluate the sentence "*about* 10 *or more students than teachers are at the party*" we can formulate the semi-fuzzy quantifier

$$Q^{T_{8,10,10,12}}_{E,3,\mathrm{card\_comp\_}\langle\langle 1,1\rangle,1\rangle}(X_1, X_2, X_3) = T_{8,10,10,12}(|X_1 \cap X_3| - |X_2 \cap X_3|)$$

where the sets X_1, X_2, X_3 represent those individuals that are students, those that are teachers, and those that are at the party, respectively.

These types of sentences can be evaluated using binary semi-fuzzy quantifiers, since [7]

$$Q^{\mathrm{FN}}_{E,3,\mathrm{card\_comp\_}\langle\langle 1,1\rangle,1\rangle}(X_1,X_2,X_3) = Q^{\mathrm{FN}}_{E,2,\mathrm{card\_comp}}(X_1 \cap X_3, X_2 \cap X_3)$$

Definition 11 (*Ternary cardinal comparative*, *type* $\langle 1,\langle 1,1\rangle\rangle$). A semi-fuzzy quantifier $Q : \wp(E)^3 \to \mathbf{I}$ is a ternary cardinal comparative of type $\langle 1,\langle 1,1\rangle\rangle$ if there exists a fuzzy number $\mathrm{FN} : \mathbb{Z} \to \mathbf{I}$ such that

$$Q(X_1,X_2,X_3) = \mathrm{FN}(|X_1 \cap X_2| - |X_1 \cap X_3|)$$

Let $\mathrm{FN} : \mathbb{Z} \to \mathbf{I}$ be a fuzzy number. We define

$$Q^{\mathrm{FN}}_{E,3,\mathrm{card\_comp\_}\langle 1,\langle 1,1\rangle\rangle} = \mathrm{FN}(|X_1 \cap X_2| - |X_1 \cap X_3|)$$

Sentences like "about 10 or more students are at the lake than at the party" can be evaluated by using this type of semi-fuzzy quantifier. These types of sentences can also be evaluated using binary semi-fuzzy quantifiers, since

$$Q^{\mathrm{FN}}_{E,3,\mathrm{card\_comp\_}\langle 1,\langle 1,1\rangle\rangle}(X_1,X_2,X_3) = Q^{\mathrm{FN}}_{E,2,\mathrm{card\_comp}}(X_1 \cap X_2, X_1 \cap X_3)$$

Definition 12 (*Ternary proportional comparative*, *type* $\langle\langle 1,1\rangle,1\rangle$). A semi-fuzzy quantifier $Q : \wp(E)^3 \to \mathbf{I}$ is a ternary proportional comparative semi-fuzzy quantifier of type $\langle\langle 1,1\rangle,1\rangle$ if there exists a fuzzy number $\mathrm{FN} : \mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ such that

$$Q(X_1,X_2,X_3) = \begin{cases} \mathrm{FN}\left(\dfrac{|X_1 \cap X_3|}{|X_2 \cap X_3|}\right) & (X_1 \cap X_3 \neq \emptyset) \wedge (X_2 \cap X_3 \neq \emptyset) \\ \mathrm{FN}(\infty) & (X_1 \cap X_3 \neq \emptyset) \wedge (X_2 \cap X_3 = \emptyset) \\ \mathrm{FN}(1) & (X_1 \cap X_3 = \emptyset) \wedge (X_2 \cap X_3 = \emptyset) \end{cases}$$

Let $\mathrm{FN} : \mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ be a fuzzy number. We define

[7] Although this transformation is valid for the case of semi-fuzzy quantifiers, none of the quantifier fuzzification mechanisms that are described in Appendix A guarantees its fulfillment. A counter-example of the non-fulfillment of the relation for the mechanism F^I is, simply $X_1 = X_2 = X_3 = \{0.5/e_1\}$ and $Q^{T_{-1,0,0,1}}_{E,3,\mathrm{card\_comp\_}\langle\langle 1,1\rangle,1\rangle}$. A counter-example for the quantifier fuzzification mechanism M is $X_1 = \{0.5/e_1\}$, $X_2 = X_3 = \{1/e_1\}$ and the same semi-fuzzy quantifier $Q^{T_{-1,0,0,1}}_{E,3,\mathrm{card\_comp\_}\langle\langle 1,1\rangle,1\rangle}$. The violation of this relation is important for practical applications; e.g., the users of a possible language that incorporate quantifiers would expect to obtain the same results regardless of the quantifier that is used. The other transformations that are described in this section also show the same problem.

$$Q^{\mathrm{FN}}_{E,3,\mathrm{prop\_comp\_}\langle\langle 1,1\rangle,1\rangle} = \begin{cases} \mathrm{FN}\left(\dfrac{|X_1 \cap X_3|}{|X_2 \cap X_3|}\right) & (X_1 \cap X_3 \neq \emptyset) \wedge (X_2 \cap X_3 \neq \emptyset) \\ \mathrm{FN}(\infty) & (X_1 \cap X_3 \neq \emptyset) \wedge (X_2 \cap X_3 = \emptyset) \\ \mathrm{FN}(1) & (X_1 \cap X_3 = \emptyset) \wedge (X_2 \cap X_3 = \emptyset) \end{cases}$$

Example 11. In order to evaluate the sentence "*about twice as many or more students than teachers are at the party*" the following semi-fuzzy quantifier can be defined

$$Q^{S_{1.5,2}}_{E,3,\mathrm{prop\_comp\_}\langle\langle 1,1\rangle,1\rangle}(X_1,X_2,X_3) = \begin{cases} S_{1.5,2}\left(\dfrac{|X_1 \cap X_3|}{|X_2 \cap X_3|}\right) & (X_1 \cap X_3 \neq \emptyset) \wedge (X_2 \cap X_3 \neq \emptyset) \\ 1 & (X_1 \cap X_3 \neq \emptyset) \wedge (X_2 \cap X_3 = \emptyset) \\ 0 & (X_1 \cap X_3 = \emptyset) \wedge (X_2 \cap X_3 = \emptyset) \end{cases}$$

where the sets X_1, X_2, X_3 respectively represent those individuals that are students, those that are teachers, and those that are at the party.

These types of sentences can be evaluated using binary semi-fuzzy quantifiers, since

$$Q^{\mathrm{FN}}_{E,3,\mathrm{prop\_comp\_}\langle\langle 1,1\rangle,1\rangle}(X_1,X_2,X_3) = Q^{\mathrm{FN}}_{E,2,\mathrm{prop\_comp}}(X_1 \cap X_3, X_2 \cap X_3)$$

Definition 13 (*Ternary proportional comparative*, *type* $\langle 1,\langle 1,1\rangle\rangle$). A semi-fuzzy quantifier $Q : \wp(E)^3 \to \mathbf{I}$ is a ternary proportional comparative semi-fuzzy quantifier of type $\langle 1,\langle 1,1\rangle\rangle$ $\mathrm{FN} : \mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ such that

$$Q(X_1,X_2,X_3) = \begin{cases} \mathrm{FN}\left(\dfrac{|X_1 \cap X_2|}{|X_1 \cap X_3|}\right) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_1 \cap X_3 \neq \emptyset) \\ \mathrm{FN}(\infty) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_1 \cap X_3 = \emptyset) \\ \mathrm{FN}(1) & (X_1 \cap X_2 = \emptyset) \wedge (X_1 \cap X_3 = \emptyset) \end{cases}$$

Let $\mathrm{FN} : \mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ be a fuzzy number. We define

$$Q^{\mathrm{FN}}_{E,3,\mathrm{prop\_comp\_}\langle 1,\langle 1,1\rangle\rangle}(X_1,X_2,X_3) = \begin{cases} \mathrm{FN}\left(\dfrac{|X_1 \cap X_2|}{|X_1 \cap X_3|}\right) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_1 \cap X_3 \neq \emptyset) \\ \mathrm{FN}(\infty) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_1 \cap X_3 = \emptyset) \\ \mathrm{FN}(1) & (X_1 \cap X_2 = \emptyset) \wedge (X_1 \cap X_3 = \emptyset) \end{cases}$$

Sentences like "at least about twice the number of students are at the lake than at the party" can be evaluated using this type of semi-fuzzy quantifiers. These types of sentences can also be evaluated using binary semi-fuzzy quantifiers, since

$$Q^{\mathrm{FN}}_{E,3,\mathrm{prop\_comp\_}\langle 1,\langle 1,1\rangle\rangle}(X_1,X_2,X_3) = Q^{\mathrm{FN}}_{E,2,\mathrm{prop\_comp}}(X_1 \cap X_2, X_1 \cap X_3)$$

The denotation of the determiners that are associated to the aforementioned sentences is $TYPE\langle\langle 1,1\rangle,1\rangle$ or $TYPE\langle 1,\langle 1,1\rangle\rangle$. [8]

In Section 3.5 an additional type of semi-fuzzy quantifier is considered, which is similar to the last case examined for quaternary semi-fuzzy quantifiers.

3.4. Quaternary semi-fuzzy quantifiers

The quantified sentences that are dealt with in this section are those involving two names and two predicates. The following ones are examples of this type of sentences:

"More students come to the party than teachers go to the lake".
"More than 10 times as many students come to the party than teachers go to the lake".
"The proportion of students that come to the party is twice that of teachers that go to the lake".
"There are proportionally about twice or more women cycling than men running".

Definition 14 (*Quaternary cardinal comparative*). A semi-fuzzy quantifier $Q : \wp(E)^4 \to \mathbf{I}$ is a quaternary cardinal comparative semi-fuzzy quantifier if there exists a fuzzy number $\mathrm{FN} : \mathbb{Z} \to \mathbf{I}$ such that

$$Q_E(X_1,X_2,X_3,X_4) = \mathrm{FN}(|X_1 \cap X_2| - |X_3 \cap X_4|)$$

Let $\mathrm{FN} : \mathbb{Z} \to \mathbf{I}$ be a fuzzy number. We define

$$Q^{\mathrm{FN}}_{E,4,\mathrm{card\_comp}} = \mathrm{FN}(|X_1 \cap X_2| - |X_3 \cap X_4|)$$

Example 12. In order to evaluate the sentence "*at least about* 10 *more students are at the lake than teachers are at the party*" the following semi-fuzzy quantifier can be defined

$$Q^{\mathrm{FN}}_{E,4,\mathrm{card\_comp}}(X_1,X_2,X_3,X_4) = T_{8,10,10,12}(|X_1 \cap X_2| - |X_3 \cap X_4|)$$

[8] In [17] a definition of cardinal comparative semi-fuzzy quantifiers is made which includes the absolute and proportional cases $\langle\langle 1,1\rangle,1\rangle$ (using a function $q : \{0,\dots,|E|\}^2 \to \mathbf{I}$, in which the first argument represents the cardinality of $|X_1 \cap X_3|$ and the second the cardinality of $|X_2 \cap X_3|$).

where the sets X_1, X_2, X_3, X_4 respectively represent those individuals that are students, those that are at the lake, those that are teachers, and those that are at the party.

As for the case of ternary semi-fuzzy quantifiers, these types of sentences can be evaluated by using binary semi-fuzzy quantifiers, since [9]

$$Q^{\mathrm{FN}}_{E,4,\mathrm{card\_comp}}(X_1, X_2, X_3, X_4) = Q^{\mathrm{FN}}_{E,2,\mathrm{card\_comp}}(X_1 \cap X_2, X_3 \cap X_4)$$

Definition 15 (*Quaternary proportional comparative*). We say that a semi-fuzzy quantifier $Q : \wp(E)^4 \to \mathbf{I}$ is a quaternary proportional comparative semi-fuzzy quantifier if there exists a fuzzy number $\mathrm{FN} : \mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ such that

$$Q_E(X_1, X_2, X_3, X_4) = \begin{cases} \mathrm{FN}\left(\dfrac{|X_1 \cap X_2|}{|X_3 \cap X_4|}\right) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_3 \cap X_4 \neq \emptyset) \\ \mathrm{FN}(\infty) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_3 \cap X_4 = \emptyset) \\ \mathrm{FN}(1) & (X_1 \cap X_2 = \emptyset) \wedge (X_3 \cap X_4 = \emptyset) \end{cases}$$

Let $\mathrm{FN} : \mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ be a fuzzy number. We define

$$Q^{\mathrm{FN}}_{E,4,\mathrm{prop\_comp}}(X_1, X_2, X_3, X_4) = \begin{cases} \mathrm{FN}\left(\dfrac{|X_1 \cap X_2|}{|X_3 \cap X_4|}\right) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_3 \cap X_4 \neq \emptyset) \\ \mathrm{FN}(\infty) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_3 \cap X_4 = \emptyset) \\ \mathrm{FN}(1) & (X_1 \cap X_2 = \emptyset) \wedge (X_3 \cap X_4 = \emptyset) \end{cases}$$

Example 13. In order to evaluate the sentence "*about twice as many or more students are at the lake than teachers are at the party*" the following semi-fuzzy quantifier can be defined

$$Q^{S_{1.5,2}}_{E,4,\mathrm{prop\_comp}}(X_1, X_2, X_3, X_4) = \begin{cases} S_{1.5,2}\left(\dfrac{|X_1 \cap X_2|}{|X_3 \cap X_4|}\right) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_3 \cap X_4 \neq \emptyset) \\ 1 & (X_1 \cap X_2 \neq \emptyset) \wedge (X_3 \cap X_4 = \emptyset) \\ 0 & (X_1 \cap X_2 = \emptyset) \wedge (X_3 \cap X_4 = \emptyset) \end{cases}$$

where the sets X_1, X_2, X_3, X_4 respectively represent those individuals that are students, those that are at the lake, those that are teachers, and those that are in the party.

[9] It is easy to find counter-examples in which the quantifier fuzzification mechanism F^I does not fulfill this transformation. The quantifier fuzzification mechanism M fulfills the aforementioned transformation.

These types of sentences can be evaluated by using semi-fuzzy quantifiers of arity 2, since

$$Q^{\mathrm{FN}}_{E,4,\mathrm{prop\_comp}}(X_1,X_2,X_3,X_4) = Q^{\mathrm{FN}}_{E,2,\mathrm{prop\_comp}}(X_1 \cap X_2, X_3 \cap X_4)$$

Definition 16 (*Quaternary comparative on proportions*). We say that a semi-fuzzy quantifier $Q : \wp(E)^4 \to \mathbf{I}$ is a quaternary comparative on proportions semi-fuzzy quantifier if there exists a function $\mathrm{FN} : \mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ such that

$$Q_E(X_1,X_2,X_3,X_4) = \begin{cases} \mathrm{FN}\left(\dfrac{\frac{|X_1 \cap X_2|}{|X_1|}}{\frac{|X_3 \cap X_4|}{|X_3|}}\right) & \dfrac{|X_1 \cap X_2|}{|X_1|} \neq 0 \wedge \dfrac{|X_3 \cap X_4|}{|X_3|} \neq 0 \\ \mathrm{FN}(\infty) & \dfrac{|X_1 \cap X_2|}{|X_1|} \neq 0 \wedge \dfrac{|X_3 \cap X_4|}{|X_3|} = 0 \\ v \in [0,1] & \text{otherwise} \end{cases}$$

Let $\mathrm{FN} : \mathbb{R}^+ \cup \{\infty\} \to \mathbf{I}$ be a fuzzy number. We define

$$Q^{\mathrm{FN},v}_{E,4,\mathrm{on\_prop\_comp}} = \begin{cases} \mathrm{FN}\left(\dfrac{\frac{|X_1 \cap X_2|}{|X_1|}}{\frac{|X_3 \cap X_4|}{|X_3|}}\right) & \dfrac{|X_1 \cap X_2|}{|X_1|} \neq 0 \wedge \dfrac{|X_3 \cap X_4|}{|X_3|} \neq 0 \\ \mathrm{FN}(\infty) & \dfrac{|X_1 \cap X_2|}{|X_1|} \neq 0 \wedge \dfrac{|X_3 \cap X_4|}{|X_3|} = 0 \\ v \in [0,1] & \text{otherwise} \end{cases}$$

Example 14. In order to evaluate the sentence "*there are proportionally about twice or more women cycling than men running*" we can formulate the semi-fuzzy quantifier

$$Q^{S_{1.5,2},0}_{E,4,\mathrm{on\_prop\_comp}}(X_1,X_2,X_3,X_4) = \begin{cases} S_{1.5,2}\left(\dfrac{\frac{|X_1 \cap X_2|}{|X_1|}}{\frac{|X_3 \cap X_4|}{|X_3|}}\right) & \dfrac{|X_1 \cap X_2|}{|X_1|} \neq 0 \wedge \dfrac{|X_3 \cap X_4|}{|X_3|} \neq 0 \\ 1 & \dfrac{|X_1 \cap X_2|}{|X_1|} \neq 0 \wedge \dfrac{|X_3 \cap X_4|}{|X_3|} = 0 \\ 0 & \text{otherwise} \end{cases}$$

where the sets X_1, X_2, X_3, X_4 respectively represent those individuals that are women, those that are cycling, those that are men, and those that are running.

A similar definition to the above could have been made for ternary semi-fuzzy quantifiers, for example, in order to evaluate sentences like "*there are*

Table 3
Definition of unary semi-fuzzy quantifiers

Unary quantifiers	
Semi-fuzzy quantifier	Example
Unary cardinal	
$Q^{\mathrm{FN}}_{E,1,\mathrm{card}}(X) = \mathrm{FN}(\lvert X\rvert)$	About five people are students
$Q^{\mathrm{FN}}_{E,1,\mathrm{prop}}(X) = \mathrm{FN}\left(\frac{\lvert X\rvert}{\lvert E\rvert}\right)$	About 70% or more of the people are students
$Q^{\mathrm{FN}}_{E,1,\mathrm{all\_but\_card}}(X) = \mathrm{FN}(\lvert E\setminus X\rvert)$	All except about five of the people are students
$Q^{\mathrm{FN}}_{E,1,\mathrm{all\_but\_prop}}(X) = \mathrm{FN}\left(\frac{\lvert E\setminus X\rvert}{\lvert E\rvert}\right)$	All except about 30% of the people are students

proportionally about twice or more women cycling than men cycling". Although in the previous section the corresponding definition was not carried out, this situation is considered in Section 3.5.

The denotation of the determiners that are associated to the aforementioned sentences is $TYPE\langle\langle 1,1\rangle,\langle 1,1\rangle\rangle$.

3.5. Classification summary

The classification of semi-fuzzy quantifiers that is explained in the previous sections is summarized in Tables 3–6. Different possibilities for the definition of semi-fuzzy quantifiers are considered, which is useful from a computational point of view.

4. Sentences involving nested quantifiers

In this section we will show how it is possible to evaluate sentences in which there are nested quantifiers, [10] leaving the treatment of other types of sentences for future studies. The approach of this section is still somewhat preliminary, but interesting from the perspective of applications. Possibly, the most obvious application field is fuzzy databases, although it is not difficult to imagine other fields in which there will be interest in the techniques that are set out.

Let us now consider some examples:

"Some men like all women".
"Most men like most women".
"Most men like more blond women than they do dark-haired women". [11]

[10] More specifically, some of the quantified sentences from the $AR\langle -1\rangle$ group [20].

[11] Here we are interpreting that the amount of blond women that each men likes is greater than the amount of dark-haired women that he likes.

Table 4
Definition of binary semi-fuzzy quantifiers

Binary quantifiers	
Semi-fuzzy quantifier	Example
Cardinal/absolute $Q^{\mathrm{FN}}_{E,2,\mathrm{card}}(X_1,X_2)=\mathrm{FN}(\lvert X_1\cap X_2\rvert)$	About five students are Spanish
Co-cardinal/exception $Q^{\mathrm{FN}}_{E,2,\mathrm{co\_card}}(X_1,X_2)=\mathrm{FN}(\lvert X_1\setminus X_2\rvert)$	All except about five students are Spanish
Proportional $Q^{\mathrm{FN},v}_{E,2,\mathrm{prop}}(X_1,X_2)=\begin{cases}\mathrm{FN}\left(\frac{\lvert X_1\cap X_2\rvert}{\lvert X_1\rvert}\right) & X_1\neq\emptyset\\ v\in[0,1] & X_1=\emptyset\end{cases}$	About 10% of students are Spanish
Proportional $Q^{\mathrm{FN},v}_{E,2,\mathrm{co\_prop}}(X_1,X_2)=\begin{cases}\mathrm{FN}\left(\frac{\lvert X_1\setminus X_2\rvert}{\lvert X_1\rvert}\right) & X_1\neq\emptyset\\ v\in[0,1] & X_1=\emptyset\end{cases}$	All except about 10% of students are Spanish
Cardinal comparative $Q^{\mathrm{FN}}_{E,2,\mathrm{card\_comp}}(X_1,X_2)=\mathrm{FN}(\lvert X_1\rvert-\lvert X_2\rvert)$	There are about five more men than women
Proportional comparative $Q^{\mathrm{FN}}_{E,2,\mathrm{prop\_comp}}(X_1,X_2)=\begin{cases}\mathrm{FN}\left(\frac{\lvert X_1\rvert}{\lvert X_2\rvert}\right) & X_2\neq\emptyset\\ \mathrm{FN}(\infty) & X_1\neq\emptyset\wedge X_2=\emptyset\\ \mathrm{FN}(1) & X_1=\emptyset\wedge X_2=\emptyset\end{cases}$	There are twice as many men as women

The syntactic analysis of the first example is given in Fig. 4, where "*like*" is a binary predicate (*like* $\in \wp(E^2)$) which combines with the determiner phrase "*all women*" to form the unary predicate "*like all women*". It should be noted that the denotation of "*like all women*" must be the unary predicate (set) of men that like all women.

Let us return to the example expounded in Section 2.1 "*all men walk*". In this example, the determiner "*all*" had a denotation $TYPE\langle 1,1\rangle$:

$$\mathbf{all}_E : (X_1\in\wp(E))\to(f : X_2\in\wp(E)\to\{0,1\})$$

This definition does not enable to deal with sentences such as the previous one, since in this case it is required that

$$\mathbf{all}'_E : (X_1\in\wp(E))\to(f : X_2\in\wp(E^2)\to\wp(E))$$

Table 5
Definition of ternary semi-fuzzy quantifiers

Ternary quantifiers	
Type of the semi-fuzzy quantifier	Example
Cardinal comparative, type $\langle\langle 1,1\rangle,1\rangle$ $Q^{\mathrm{FN}}_{E,3,\mathrm{card\_comp\_}\langle\langle 1,1\rangle,1\rangle}(X_1,X_2,X_3) = \mathrm{FN}(\lvert X_1\cap X_3\rvert - \lvert X_2\cap X_3\rvert)$	About 10 or more students than teachers are at the party
Cardinal comparative, type $\langle 1,\langle 1,1\rangle\rangle$ $Q^{\mathrm{FN}}_{E,3,\mathrm{card\_comp\_}\langle 1,\langle 1,1\rangle\rangle}(X_1,X_2,X_3) = \mathrm{FN}(\lvert X_1\cap X_2\rvert - \lvert X_1\cap X_3\rvert)$	About 10 or more students are at the lake than at the party
Proportional comparative, type $\langle\langle 1,1\rangle,1\rangle$ $Q^{\mathrm{FN}}_{E,3,\mathrm{card\_prop\_}\langle\langle 1,1\rangle,1\rangle}(X_1,X_2,X_3) = \begin{cases} \mathrm{FN}\left(\frac{\lvert X_1\cap X_3\rvert}{\lvert X_2\cap X_3\rvert}\right) & (X_1\cap X_3\neq\emptyset)\wedge(X_2\cap X_3\neq\emptyset)\\ \mathrm{FN}(\infty) & (X_1\cap X_3\neq\emptyset)\wedge(X_2\cap X_3=\emptyset)\\ \mathrm{FN}(1) & (X_1\cap X_3=\emptyset)\wedge(X_2\cap X_3=\emptyset)\end{cases}$	About twice as many or more students than teachers are at the party
Proportional comparative, type $\langle 1,\langle 1,1\rangle\rangle$ $Q^{\mathrm{FN}}_{E,3,\mathrm{card\_prop\_}\langle 1,\langle 1,1\rangle\rangle}(X_1,X_2,X_3) = \begin{cases} \mathrm{FN}\left(\frac{\lvert X_1\cap X_2\rvert}{\lvert X_1\cap X_3\rvert}\right) & (X_1\cap X_2\neq\emptyset)\wedge(X_1\cap X_3\neq\emptyset)\\ \mathrm{FN}(\infty) & (X_1\cap X_2\neq\emptyset)\wedge(X_1\cap X_3=\emptyset)\\ \mathrm{FN}(1) & (X_1\cap X_2=\emptyset)\wedge(X_1\cap X_3=\emptyset)\end{cases}$	At least about twice the number of students are at the lake than at the party
Comparative on proportions, type $\langle\langle 1,1\rangle,1\rangle$ $Q^{\mathrm{FN},v}_{E,3,\mathrm{comp\_on\_prop\_}\langle\langle 1,1\rangle,1\rangle}(X_1,X_2,X_3) = \begin{cases} \mathrm{FN}\left(\frac{\frac{\lvert X_1\cap X_3\rvert}{\lvert X_1\rvert}}{\frac{\lvert X_2\cap X_3\rvert}{\lvert X_2\rvert}}\right) & \frac{\lvert X_1\cap X_3\rvert}{\lvert X_1\rvert}\neq 0\wedge\frac{\lvert X_2\cap X_3\rvert}{\lvert X_2\rvert}\neq 0\\ \mathrm{FN}(\infty) & \frac{\lvert X_1\cap X_3\rvert}{\lvert X_1\rvert}\neq 0\wedge\frac{\lvert X_2\cap X_3\rvert}{\lvert X_2\rvert}= 0\\ v\in[0,1] & \text{otherwise}\end{cases}$	There are proportionally about twice or more women cycling than men cycling
Proportional comparative, type $\langle 1,\langle 1,1\rangle\rangle$ $Q^{\mathrm{FN}}_{E,3,\mathrm{comp\_on\_prop\_}\langle 1,\langle 1,1\rangle\rangle}(X_1,X_2,X_3) = \mathrm{FN}\left(\frac{\frac{\lvert X_1\cap X_2\rvert}{\lvert X_1\rvert}}{\frac{\lvert X_1\cap X_3\rvert}{\lvert X_1\rvert}}\right) = \mathrm{FN}\left(\frac{\lvert X_1\cap X_2\rvert}{\lvert X_1\cap X_3\rvert}\right)$	There are proportionally about twice or more women cycling than women running

Table 6
Definition of quaternary semi-fuzzy quantifiers

Quaternary quantifiers	
Type of the semi-fuzzy quantifier	Example
Cardinal comparative $Q^{\mathrm{FN}}_{E,4,\mathrm{card\_comp}}(X_1 X_2, X_3, X_4) = \mathrm{FN}(\lvert X_1 \cap X_2\rvert - \lvert X_3 \cap X_4\rvert)$	At least about 10 more students are at the lake than teachers are at the party
Cardinal proportional comparative $Q^{\mathrm{FN}}_{E,4,\mathrm{prop\_comp}}(X_1 X_2, X_3, X_4) = \begin{cases} \mathrm{FN}\left(\frac{\lvert X_1 \cap X_2\rvert}{\lvert X_3 \cap X_4\rvert}\right) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_3 \cap X_4 \neq \emptyset) \\ \mathrm{FN}(\infty) & (X_1 \cap X_2 \neq \emptyset) \wedge (X_3 \cap X_4 = \emptyset) \\ \mathrm{FN}(1) & (X_1 \cap X_2 = \emptyset) \wedge (X_3 \cap X_4 = \emptyset) \end{cases}$	About twice as many or more students are at the lake than teachers are at the party
Comparative on proportions $Q^{\mathrm{FN},v}_{E,4,\mathrm{on\_prop\_comp}}(X_1 X_2, X_3, X_4) = \begin{cases} \mathrm{FN}\left(\frac{\frac{\lvert X_1 \cap X_2\rvert}{\lvert X_1\rvert}}{\frac{\lvert X_3 \cap X_4\rvert}{\lvert X_3\rvert}}\right) & \frac{\lvert X_1 \cap X_2\rvert}{\lvert X_1\rvert} \neq 0 \wedge \frac{\lvert X_3 \cap X_4\rvert}{\lvert X_3\rvert} \neq 0 \\ \mathrm{FN}(\infty) & \frac{\lvert X_1 \cap X_2\rvert}{\lvert X_1\rvert} \neq 0 \wedge \frac{\lvert X_3 \cap X_4\rvert}{\lvert X_3\rvert} = 0 \\ v \in [0,1] & \text{otherwise} \end{cases}$	There are proportionally about twice or more women cycling than men running

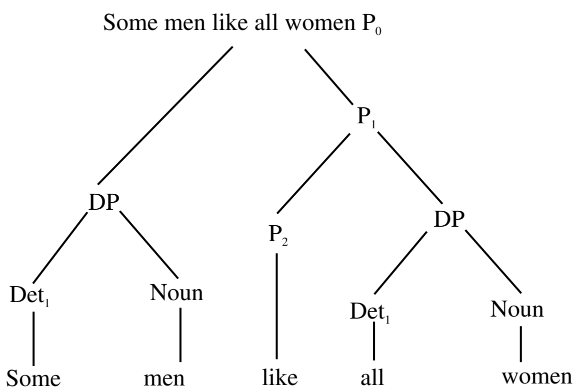

Fig. 4. Syntactic analysis of the quantified sentence “some men like all women”.

i.e., $\mathbf{all}'_E$ is a function that takes a property as an argument (in the example above, “*women*”) and returns a function with a domain in the set of binary relations, and range in the set of unary relations or properties.

At least the following two denotations for **all** are needed:

$$\mathbf{all}_{1E} : (X_1 \in \wp(E)) \to (f : X_2 \in \wp(E) \to \{0,1\})$$
$$\mathbf{all}_{2E} : (X_1 \in \wp(E)) \to (f : X_2 \in \wp(E^2) \to \wp(E))$$

The second example "*most men like most women*" is similar to the first one. Let us suppose that in the third example, "*most men like more blond women than they do dark-haired women*" properties "*blond women*" and "*dark-haired women*" are crisp. In principle **more** ... **than** should be a *TYPE*$\langle\langle 1,1\rangle,1\rangle$ function, but what we really require now is

$$(\mathbf{more}\ldots\mathbf{than})'_E : (X_1, X_2 \in \wp(E)) \to (f : X_3 \in \wp(E^2) \to \wp(E))$$

i.e., we need at least the following two denotations for **more** ... **than**:

$$(\mathbf{more}\ldots\mathbf{than})_{1E} : (X_1, X_2 \in \wp(E)) \to (f : X_3 \in \wp(E) \to \{0,1\})$$
$$(\mathbf{more}\ldots\mathbf{than})_{2E} : (X_1, X_2 \in \wp(E)) \to (f : X_3 \in \wp(E^2) \to \wp(E))$$

Here it should be noted that the determiners appearing in the above examples were classified as *TYPE*$\langle \_, 1\rangle$; i.e., the determiner phrases of quantified sentences always take a unique predicate (not 2 as in the examples given in Section 3.3 and Section 3.4). This fact will restrict the type of expressions that we need to model.

We now go on to make a number of definitions that make it possible to resolve situations of this type. In [20] the treatment of expressions of this type is dealt with at length. The modus operandi that we use is aimed at the future definition of a language that will make it possible to evaluate fuzzy quantified sentences.

Firstly, we introduce some notation:

Definition 17. Let $r > 0$, $R \in \wp(E^{r+1})$ a crisp $(r+1)$-ary relation over E and $x = \langle x_1, \ldots, x_r\rangle$ an n-tuple of elements of E. We denote

$$xR = \{b \in E : \langle x_1, \ldots, x_r, b\rangle \in R\}$$

Example 15. Let us consider the following crisp relation "*like*" $\in \wp(E^2)$:

$$Like = \left\{ \begin{array}{l} (John, Mary), (John, Eve), (Sonya, Robert), \\ (Sonya, John), (John, Sonya), (Robert, Sonya) \end{array} \right\}$$

thus the set of people that John likes is:

$$\langle John\rangle Like = \{Mary, Eve, Sonya\}$$

The following notation is similar to the one above, but it permits "selection" by arbitrary arguments:

Definition 18. Let $r > 0$, $R \in \wp(E^{r+1})$ a crisp $(r+1)$-ary relation over E and $x = \langle x_1, \ldots, x_r \rangle$ an r-tuple of elements of E we denote

$$x_j R = \{b \in E : \langle x_1, \ldots, x_{j-1}, b, x_j, \ldots, x_r \rangle \in R\}, \quad 1 \leqslant j \leqslant r+1$$

Example 16. In the previous example the set of people that John likes is

$$\langle John \rangle_2 Like = \{Mary, Eve, Sonya\}$$

and the set of people that like Sonya is

$$\langle Sonya \rangle_1 Like = \{John, Robert\}$$

In the case of R being a fuzzy relation, the previous definition is transformed into the following one:

Definition 19. Let $r > 0$, $R \in \widetilde{\wp}(E^{r+1})$ a fuzzy $(r+1)$-ary relation over E and $x = \langle x_1, \ldots, x_r \rangle$ an r-tuple of elements of E (it should be noted that x is crisp), we denote as $x_j R$, $1 \leqslant j \leqslant r+1$ the fuzzy set over E such that

$$\mu_{x_j R}(b) = \mu_R(\langle x_1, \ldots, x_{j-1}, b, x_j, \ldots, x_r \rangle)$$

Example 17. Let us consider a fuzzy relation "*like*" $\in \widetilde{\wp}(E^2)$, which is denoted as follows:

$$Like = \left\{ \begin{array}{l} 0.8/(John, Mary), 1/(John, Eve), 0.5/(Sonya, Robert), \\ 1/(Sonya, John), 0.2/(John, Sonya) \end{array} \right\}$$

then

$$\langle John \rangle_2 Like = \{0.8/Mary, 1/Eve, 0.2/Sonya\}$$

Definition 20 (*Operator* $\%^i_Q$ *on tuples*). Let $\langle x_1, \ldots, x_r \rangle \in E^r$, $R \in \wp(E^{r+1})$ be a crisp $(r+1)$-ary relation. We define the operator on tuples $\%^i_Q : E^r \times \wp(E^{r+1}) \to \mathbf{I}$, $1 \leqslant j \leqslant r+1$ dependent on the semi-fuzzy quantifier $Q : \wp(E) \to \mathbf{I}$, and which we use in infix notation as

$$\langle x_1, \ldots, x_r \rangle \%^i_Q R = Q(\langle x_1, \ldots, x_r \rangle_i R)$$

Example 18. Let

$$\begin{aligned} men &= \{John, Robert, Peter\} \\ women &= \{Mary, Eve, Sonya, Esther\} \\ Like &= \left\{ \begin{array}{l} (John, Mary), (John, Eve), (Sonya, Robert), \\ (Robert, Sonya), (Sonya, John), (John, Sonya) \end{array} \right\} \end{aligned}$$

and the semi-fuzzy quantifier

$$\mathbf{nearly\_all\_women}_E(X) = S_{0.5,1}\left(\frac{|X \cap women|}{|women|}\right)$$

then,

$$\begin{aligned}\langle John\rangle\%^2_{\mathbf{nearly\_all\_women}_E} Like &= \mathbf{nearly\_all\_women}_E(\langle John\rangle_2 Like)\\ &= \mathbf{nearly\_all\_women}_E(\{Mary, Eve, Sonya\})\\ &= 0.5\end{aligned}$$

the meaning of which is that "John likes nearly all women is true to the degree of 0.5", and

$$\begin{aligned}\langle Robert\rangle\%^2_{\mathbf{nearly\_all\_women}_E} Like &= \mathbf{nearly\_all\_women}_E(\langle Robert\rangle_2 Like)\\ &= \mathbf{nearly\_all\_women}_E(\{Sonya\})\\ &= 0\end{aligned}$$

$$\begin{aligned}\langle Peter\rangle\%^2_{\mathbf{nearly\_all\_women}_E} Like &= \mathbf{nearly\_all\_women}_E(\langle Peter\rangle Like)\\ &= \mathbf{nearly\_all\_women}_E(\emptyset)\\ &= 0\end{aligned}$$

Definition 21 (*Operator $\%^i_Q$ on sets*). Let $R \in \wp(E^{r+1})$ be a crisp $(r+1)$-ary relation, and $Q : \wp(E) \to \mathbf{I}$ be a semi-fuzzy quantifier of arity 1. The operator on sets $\%^i_Q : \wp(E^{r+1}) \to \widetilde{\wp}(E^r)$ which is dependent on the semi-fuzzy quantifier $Q : \wp(E) \to \mathbf{I}$ have as an image the fuzzy set $\%^i_Q(R)$, $1 \leqslant j \leqslant r+1$ whose membership function is:

$$\begin{aligned}\mu_{\%^i_Q(R)}(\langle x_1, \ldots, x_r\rangle) &= \langle x_1, \ldots, x_r\rangle\%^i_Q R\\ &= Q(\langle x_1, \ldots, x_r\rangle_i R)\end{aligned}$$

where $\langle x_1, \ldots, x_r\rangle \in E^r$.

Example 19. For the previous example we have

$$\%^2_{\mathbf{nearly\_all\_women}_E}(like) = \{0.5/John\}$$

i.e., $\%^i_{\mathbf{nearly\_all\_women}_E}(like)$ is the set of people who like nearly all women.

Here it is worthwhile reconsidering the explanation given at the start of this section in order to understand better the definitions that have just been given. The examples given at the start of the section were:

"Some men like all women".
"Most men like most women".
"Most men like more blond women than they do dark-haired women".

Here, determiners "*all*" and "*most*" are associated to $TYPE\langle 1,1\rangle$ in Section 3, whilst "*more than*" is associated with a denotation of $TYPE\langle\langle 1,1\rangle,1\rangle$. As has already been mentioned, these determiners combine with one single predicate.

Let us now consider the combination of these determiners with nominal phrases, and leave aside, for the moment, the appearance of predicates that denote arity relations higher than 1 in quantified sentences of this type:

$$\mathbf{all}_E(women) : \wp(E) \to \mathbf{I}$$

$$\mathbf{most}_E(women) : \wp(E) \to \mathbf{I}$$

$$\mathbf{more\_than}_E(blond\_women, dark\_women) : \wp(E) \to \mathbf{I}$$

Once the function that denotes the semantics of the determiners is combined with the nominal phrases, we are left with the functions $f : \wp(E) \to \mathbf{I}$. Hence, the definition of $\%^i_Q$ is dependent on a semi-fuzzy quantifier $Q : \wp(E) \to \mathbf{I}$.

In this manner, in order to formulate the evaluation of the aforementioned quantified sentences we resolve that:

$$\mathbf{some}_E(men)\ \left(\%^2_{\mathbf{all}_E(women)}(like)\right)$$

$$\mathbf{most}_E(men)\ \left(\%^2_{\mathbf{most}_E(women)}(like)\right)$$

$$\mathbf{most}_E(men)\ \left(\%^2_{\mathbf{more}\ldots\mathbf{than}(blond\_women,dark\_women)}(like)\right)$$

The analogue of the above definition for fuzzy quantifiers is:

Definition 22 (*Operator* $\widetilde{\%^i_{\widetilde{Q}}}$ *on tuples*). Let $\langle x_1,\ldots,x_r\rangle \in E^r$, $R \in \widetilde{\wp}(E^{r+1})$ be a fuzzy relation $(r+1)$-ary. We define the operator on tuples $\widetilde{\%}^i_{\widetilde{Q}} : E^r \times \widetilde{\wp}(E^{r+1}) \to \mathbf{I}$, $1 \leqslant j \leqslant r+1$ dependent on the fuzzy quantifier $\widetilde{Q} : \widetilde{\wp}(E) \to \mathbf{I}$, and which we will use in infix notation as

$$\langle x_1,\ldots,x_r\rangle \widetilde{\%}^i_{\widetilde{Q}} R = \widetilde{Q}(\langle x_1,\ldots,x_r\rangle_i R)$$

The definition of $\widetilde{Q}$ is made on the basis of some quantifier fuzzification mechanism (see Appendix A).

Example 20. Let

$$men = \{John, Robert, Peter\}$$

$$blond\_women = \{1/Mary, 0/Eve, 0/Sonya, 0.5/Esther\}$$

$$Like = \left\{\begin{array}{l} 0.8/(John, Mary), 1/(John, Eve), 0.5/(Sonya, Robert), \\ 1/(Sonya, John), 0.2/(John, Sonya) \end{array}\right\}$$

Let the semi-fuzzy quantifier "nearly all" be defined as:

$$\mathbf{nall}_E(X_1, X_2) = \begin{cases} S_{0.5,1}\left(\frac{|X_1 \cap X_2|}{|X_1|}\right) & X_1 \neq \emptyset \\ 1 & X_1 = \emptyset \end{cases}$$

The fuzzy quantifier resulting from the application of the mechanism F^I (see Appendix A) to $\mathbf{nall}_E$ is

$$\widetilde{\mathbf{nall}}_E(X_1, X_2) = \int_0^1 \int_0^1 S_{0.5,1}\left(\frac{|(X_1)_{\geqslant \alpha_1} \cap (X_2)_{\geqslant \alpha_2}|}{|(X_1)_{\geqslant \alpha_1}|}\right) \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2$$

and

$$\begin{aligned}
&\widetilde{\mathbf{nall}}_E(blond_women)(X_2) \\
&\quad = \int_0^1 \int_0^1 S_{0.5,1}\left(\frac{|(blond_women)_{\geqslant \alpha_1} \cap (X_2)_{\geqslant \alpha_2}|}{|(blond_women)_{\geqslant \alpha_1}|}\right) \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2
\end{aligned}$$

with the meaning of nearly all blond women. Thus,

$$\begin{aligned}
&\langle John \rangle \widetilde{\%}^2_{\widetilde{\mathbf{nall}}_E(blond_women)} Like \\
&\quad = \widetilde{\mathbf{nall}}_E(blond_women)(\langle John \rangle_2 Like) \\
&\quad = \widetilde{\mathbf{nall}}_E(blond_women)(\{0.8/Mary, 1/Eve, 0.2/Sonya\}) \\
&\quad = \int_0^1 \int_0^1 S_{0.5,1}\left(\frac{|blond_women_{\geqslant \alpha_1} \cap \{0.8/Mary, 1/Eve, 0.2/Sonya\}_{\geqslant \alpha_2}|}{|blond_women_{\geqslant \alpha_1}|}\right) \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \\
&\quad = \int_0^1 \int_0^1 S_{0.5,1}\left(\frac{\left|\begin{array}{l} \{1/Mary, 0/Eve, 0/Sonya, 0.5/Esther\}_{\geqslant \alpha_1} \cap \\ \{0.8/Mary, 1/Eve, 0.2/Sonya\}_{\geqslant \alpha_2} \end{array}\right|}{|\{1/Mary, 0/Eve, 0/Sonya, 0.5/Esther\}_{\geqslant \alpha_1}|}\right) \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \\
&\quad = 0.4
\end{aligned}$$

Conversely, when applying the mechanism M (see Appendix A) on $\mathbf{nall}_E$ we obtain the fuzzy quantifier

$$\widetilde{\mathbf{nall}}_E(X_1, X_2) = \int_0^1 (S_{0.5,1})_\gamma(X_1, X_2) \, \mathrm{d}\gamma$$

and

$$\widetilde{\mathbf{nall}}_E(blond\_women)(X_2) = \int_0^1 (S_{0.5,1})_\gamma(blond\_women, X_2) \, \mathrm{d}\gamma$$

Thus,

$$
\begin{aligned}
&\langle John\rangle \widetilde{\%}^{2}_{\widetilde{\mathbf{nall}_E(blond\_women)}} Like \\
&\quad = \widetilde{\mathbf{nall}}_E(blond\_women)(\langle John\rangle_2 Like) \\
&\quad = \widetilde{\mathbf{nall}}_E(blond\_women)(\{0.8/Mary, 1/Eve, 0.2/Sonya\}) \\
&\quad = \int_0^1 (S_{0.5,1})_\gamma \begin{pmatrix} \{1/Mary, 0/Eve, 0/Sonya, 0.5/Esther\}, \\ \{0.8/Mary, 1/Eve, 0.2/Sonya\} \end{pmatrix} \mathrm{d}\gamma \\
&\quad = 0.5
\end{aligned}
$$

Definition 23 (*Operator* $\widetilde{\%}^{i}_{\widetilde{Q}}$ *on sets*). Let $R \in \widetilde{\wp}(E^{r+1})$ be a fuzzy relation $(r+1)$-ary, and $\widetilde{Q} : \widetilde{\wp}(E) \to \mathbf{I}$ a fuzzy quantifier of arity 1. The operator on sets $\widetilde{\%}^{i}_{\widetilde{Q}} : \widetilde{\wp}(E^{r+1}) \to \widetilde{\wp}(E^{r})$, $1 \leqslant j \leqslant r+1$ dependent on the quantifier $\widetilde{Q} : \widetilde{\wp}(E) \to \mathbf{I}$ has as an image the fuzzy set $\widetilde{\%}^{i}_{\widetilde{Q}}(R)$ whose membership function is:

$$
\begin{aligned}
\mu_{\widetilde{\%}^{i}_{\widetilde{Q}}}(\langle x_1, \ldots, x_r\rangle) &= \langle x_1, \ldots, x_r\rangle \widetilde{\%}^{i}_{\widetilde{Q}} R \\
&= \widetilde{Q}(\langle x_1, \ldots, x_r\rangle_i R)
\end{aligned}
$$

where $\langle x_1, \ldots, x_r\rangle \in E^r$.

Example 21. For the above example (when we use the mechanism F^I) we have

$$
\widetilde{\%}^{i}_{\widetilde{\mathbf{nall}_E(blond\_women)}}(like) = \{0.4/John, 0/Peter, 0/Robert\}
$$

With the above operations, we can now formulate the evaluation of quantified sentences similar to the ones from the start of the section:

Example 22. Let us suppose the evaluation of the sentence

some dark-haired men like nearly all blond women

where

$$dark\_haired\_men = \{1/John, 0.6/Robert, 0/Peter\}$$

$$blond\_women = \{1/Mary, 0/Eve, 0/Sonya, 0.5/Esther\}$$

$$
Like = \left\{ \begin{array}{l} 0.8/(John, Mary), 1/(John, Eve), 0.5/(Sonya, Robert), \\ 1/(Sonya, John), 0.2/(John, Sonya) \end{array} \right\}
$$

and the semi-fuzzy quantifiers $\mathbf{some}_E$ (some) and $\mathbf{nall}_E$ (nearly all) are given by

$$\mathbf{some}_E(X_1, X_2) = \begin{cases} 0 & |X_1 \cap X_2| = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\mathbf{nall}_E(X_1, X_2) = \begin{cases} S_{0.5,1}\left(\dfrac{|X_1 \cap X_2|}{|X_1|}\right) & X_1 \neq \emptyset \\ 1 & X_1 = \emptyset \end{cases}$$

The fuzzy version of the semi-fuzzy quantifiers given above, using the mechanism F^I is (see Appendix A)

$$\widetilde{\mathbf{some}}_E(X_1, X_2) = \int_0^1 \int_0^1 \mathbf{some}_E((X_1)_{\geqslant \alpha_1}, (X_2)_{\geqslant \alpha_2})\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2, X_1, X_2 \in \widetilde{\wp}(E)$$

$$\widetilde{\mathbf{nall}}_E(X_1, X_2) = \int_0^1 \int_0^1 \mathbf{nall}(_E(X_1)_{\geqslant \alpha_1}, (X_2)_{\geqslant \alpha_2})\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2, X_1, X_2 \in \widetilde{\wp}(E)$$

and the evaluation of the above sentence is formulated as follows:

$$\widetilde{\mathbf{some}}_E\left(dark\_haired\_men, \widetilde{\%}^2_{\widetilde{\mathbf{nall}}_E(blond\_women)}(like)\right)$$

In the previous example it was seen that

$$\widetilde{\%}^2_{\widetilde{\mathbf{nall}}_E(blond\_women)}(like) = \{0.4/John\}$$

Thus,

$$\begin{aligned} &\widetilde{\mathbf{some}}_E\left(dark\_haired\_men, \widetilde{\%}^2_{\widetilde{\mathbf{nall}}(blond\_women)}(like)\right) \\ &\quad = \widetilde{\mathbf{some}}_E(dark\_haired\_men, \{0.4/John\}) \\ &\quad = \widetilde{\mathbf{some}}_E(\{1/John, 0.6/Robert, 0/Peter\}, \{0.4/John\}) \\ &\quad = 0.4 \end{aligned}$$

5. Discussion

The operations explained in this paper make it possible to evaluate many of the quantified sentences that can be encountered in natural language. The approach that has been followed [14,16,18], based on the theory of generalized quantifiers [3,19–21,26], enables us to model in a simple manner not only those sentences that are habitually considered [29], but also other groups of quantified sentences, such as comparative and exception sentences, or those involving nested quantifiers.

Although, in our opinion, the cases that are contemplated are of greater practical interest, there are a number of situations that are not considered in the present work; for example non-quantitative sentences [20], such as:

"everybody except John is at the party",
"all the men except John are at the party",
"John and Mary are at the party"

have not been dealt with.

By way of an example, the evaluation of the first sentence can be carried out by means of the semi-fuzzy quantifier:

$$\textbf{all\_except\_john}_E(X) = \begin{cases} 0 & E \setminus X \neq \{John\} \\ 1 & E \setminus X = \{John\} \end{cases}$$

Generally, it is not difficult to find the mathematical definition for the semi-fuzzy quantifiers that are required to evaluate non-quantitative sentences. The problem seems to be rather the definition of a language that would make it possible to deal with these expressions in a simple manner. In order to consider these sentences it would seem to advisable to use a scheme similar to that of fuzzy relational databases, extending the fuzzy SQL with fuzzy quantifiers [4,5].

But besides non-quantitative sentences, there are other types of sentences that cannot be dealt with using the techniques that have been outlined. The following examples are taken from [20]:

"Every man danced with every women except Hans with Mary".
"Mary praised every student but herself".
"Every student answered different questions".
"Quite a few of the boys in my class and most of the girls in your class have all dated each other".
"Most students know more girls than every teacher".

6. Conclusions

Throughout this work we defined and classified a wide number of semi-fuzzy quantifiers [14,16,18] which enable the evaluation of the most important groups of quantified sentences from a practical perspective. There are a number of different models for extending these semi-fuzzy quantifiers to fuzzy quantifiers [10–12,14,16,18]. In this way, evaluation of fuzzy quantified sentences only requires selecting an appropriate fuzzy quantifier mechanism.

By taking into account the classic linguistic theories [3,19–21,26], which are more extensively developed than those of fuzzy quantification, the classification proposed in [29] is significantly expanded, and therefore show quantifier fuzzification mechanisms to be a very powerful tool for modelling fuzzy quantified sentences.

From a practical point of view, the definition and the classification in the present work make it possible to construct applications with higher than usual capabilities for handling quantified sentences, which is highly important in fields like fuzzy expert systems, fuzzy temporal knowledge representation and reasoning, natural language processing, data mining, etc. Future studies will be aimed at including sentences involving other types of quantifiers, such as non-quantitative ones.

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Appendix A

We describe below two mechanisms defined in the literature that enable the definition of fuzzy quantifiers on the basis of semi-fuzzy quantifiers. More in-depth explanations of these models, along with definitions of other alternatives, can be found in [8,10,12,14,16,18].

A.1. Mechanism M

Mechanism M for the fuzzification of semi-fuzzy quantifiers is defined in [14,16,18]. This mechanism fulfills the axiomatic framework that is defined by the author, guaranteeing a highly favourable behaviour.

Definition 24. Suppose E is some set, $X \in \widetilde{\wp}(E)$ and $\gamma \in \mathbf{I}$. $X_\gamma^{\min}, X_\gamma^{\max} \in \wp(E)$ are defined by

$$X_\gamma^{\min} = \begin{cases} X_{>\frac{1}{2}} & \gamma = 0 \\ X_{\geqslant \frac{1}{2}+\frac{1}{2}\gamma} & \gamma > 0 \end{cases}$$

$$X_\gamma^{\max} = \begin{cases} X_{\geqslant \frac{1}{2}} & \gamma = 0 \\ X_{>\frac{1}{2}-\frac{1}{2}\gamma} & \gamma > 0 \end{cases}$$

where $X_{\geqslant\alpha} = \{e \in E : \mu_X(e) \geqslant \alpha\}$ is α-cut and $X_{>\alpha} = \{e \in E : \mu_X(e) > \alpha\}$ is strict α-cut.

Definition 25 (*Fuzzy median*). The fuzzy median $med_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \to \mathbf{I}$ is defined by

$$med_{\frac{1}{2}}(u_1, u_2) = \begin{cases} \min(u_1, u_2) & \min(u_1, u_2) > \frac{1}{2} \\ \max(u_1, u_2) & \max(u_1, u_2) < \frac{1}{2} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Definition 26. The generalised fuzzy median $m_{\frac{1}{2}} : \wp(\mathbf{I}) \to \mathbf{I}$ is defined by

$$m_{\frac{1}{2}}X = med_{\frac{1}{2}}(\inf X, \sup X)$$

for all $X \in \wp(\mathbf{I})$.

Definition 27 ((Q_γ) [14,16]). Fuzzy quantifier $Q_\gamma : \widetilde{\wp}(E)^s \to \mathbf{I}$ is defined by

$$Q_\gamma(X_1, \ldots, X_s) = m_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_s) : (X_i)_\gamma^{\min} \subseteq Y_i \subseteq (X_i)_\gamma^{\max}\}$$

for all semi-fuzzy quantifiers $Q : \wp(E)^s \to \mathbf{I}$.

Definition 28 ((M) [14,16]). For every semi-fuzzy quantifier $Q : \wp(E)^s \to \mathbf{I}$, the fuzzy quantifier $M(Q) : \widetilde{\wp}(E)^s \to \mathbf{I}$ is defined by

$$M(Q)(X_1, \ldots, X_s) = \int_0^1 Q_\gamma(X_1, \ldots, X_s)\,\mathrm{d}\gamma$$

Example 23. Let us consider the sentence

almost all tall women are blond

where the semi-fuzzy quantifier $\mathbf{almost\_all}_E$, and the fuzzy sets *tall* and *blond* take the following values:

$$tall = \{0.8/e_1, 0.9/e_2, 1/e_3, 0.2/e_4\}$$

$$blond = \{1/e_1, 0.8/e_2, 0.3/e_3, 0.1/e_4\}$$

$$\mathbf{almost\_all}_E(X_1, X_2) = \begin{cases} \max\left\{2\left(\dfrac{|X_1 \cap X_2|}{|X_1|}\right) - 1, 0\right\} & X_1 \neq \emptyset \\ 1 & X_1 = \emptyset \end{cases}$$

Table 7 shows the calculation of $(tall)_\gamma^{\min}$, $(tall)_\gamma^{\max}$, $(blond)_\gamma^{\min}$ and $(blond)_\gamma^{\max}$ for the different values of γ; and Table 8 shows the calculation of Q_γ (see expression (49)). In this way we obtain

$$M(\mathbf{almost\_all}_E)(X_1, X_2) = \tfrac{1}{3} \times 0.4 + \tfrac{1}{2} \times 0.6 = 0.433$$

Table 7
Calculation of $(tall)_\gamma^{\min}$, $(tall)_\gamma^{\max}$, $(blond)_\gamma^{\min}$ and $(blond)_\gamma^{\max}$

	$(tall)_\gamma^{\min}$	$(tall)_\gamma^{\max}$	$(blond)_\gamma^{\min}$	$(blond)_\gamma^{\max}$
$\gamma \in [0, 0.4]$	$\{e_1, e_2, e_3\}$	$\{e_1, e_2, e_3\}$	$\{e_1, e_2\}$	$\{e_1, e_2\}$
$\gamma \in (0.4, 0.6]$	$\{e_1, e_2, e_3\}$	$\{e_1, e_2, e_3\}$	$\{e_1, e_2\}$	$\{e_1, e_2, e_3\}$
$\gamma \in (0.6, 0.8]$	$\{e_2, e_3\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_1\}$	$\{e_1, e_2, e_3\}$
$\gamma \in (0.8, 1]$	$\{e_3\}$	$\{e_1, e_2, e_3, e_4\}$	$\{e_1\}$	$\{e_1, e_2, e_3, e_4\}$

Table 8
Calculation of $(\mathbf{almost\_all}_E)_\gamma(X_1, X_2)$

$(\mathbf{almost\_all}_E)_\gamma(X_1, X_2)$	
$\gamma \in (0, 0.4]$	$m_{\frac{1}{2}}(Q(\{e_1, e_2, e_3\}, \{e_1, e_2\})) = \frac{1}{3}$
$\gamma \in (0.4, 0.6]$	$m_{\frac{1}{2}}(Q(\{e_1, e_2, e_3\}, \{e_1, e_2\}), Q(\{e_1, e_2, e_3\}, \{e_1, e_2, e_3\})) = \frac{1}{2}$
$\gamma \in (0.6, 0.8]$	$\frac{1}{2}$
$\gamma \in (0.8, 1]$	$\frac{1}{2}$

A.2. Mechanism F^I

According to the formulation in [9], probabilistic mechanism F^I is defined as:

$$F^I(Q)(X_1, \ldots, X_s) = \int_0^1 \cdots \int_0^1 Q((X_1)_{\geqslant \alpha_1}, \ldots, (X_s)_{\geqslant \alpha_s})\, \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_s \qquad (9)$$

where X_s, $s = 1, \ldots, S \in \widetilde{\wp}(E)$ are fuzzy properties; $(X_s)_{\geqslant \alpha_s}$ denotes an α-cut level α_s of X_s; and Q is a semi-fuzzy quantifier of arity S. Mechanism F^I cannot be considered as a quantifier fuzzification mechanism in the sense of definition (5), since the integral may not exist for non-finite referentials. [12]

[12] For example, let $Q(X)$ the semi-fuzzy quantifier on $\wp([0, 1])$ defined as

$$Q(X) = \begin{cases} 0 & \inf\ (X) \in \mathbb{Q} \\ 1 & \inf\ (X) \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

and let $X \in \widetilde{\wp}([0, 1])$ defined as

$$\mu_X(x) = x, \quad x \in [0, 1]$$

then

$$F^I(Q)(X) = \int_0^1 Q(X_{\geqslant \alpha})\, \mathrm{d}\alpha = \int_0^1 Q([\alpha, 1])\, \mathrm{d}\alpha = \int_0^1 \begin{cases} 0 & \alpha \in \mathbb{Q} \\ 1 & \alpha \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \mathrm{d}\alpha$$

and the upper integral is different from the lower integral.

Table 9
Calculation of the α-cuts of *tall* and *blond*

	$(tall)_{\geqslant \alpha_1}$		$(blond)_{\geqslant \alpha_2}$
$\alpha_1 \in (0.9, 1]$	$\{e_3\}$	$\alpha_2 \in [1, 0.8]$	$\{e_1\}$
$\alpha_1 \in (0.8, 0.9]$	$\{e_2, e_3\}$	$\alpha_2 \in (0.8, 0.3]$	$\{e_1, e_2\}$
$\alpha_1 \in (0.2, 0.8]$	$\{e_1, e_2, e_3\}$	$\alpha_2 \in (0.3, 0.1]$	$\{e_1, e_2, e_3\}$
$\alpha_1 \in (0, 0.2]$	$\{e_1, e_2, e_3, e_4\}$	$\alpha_2 \in (0.1, 0]$	$\{e_1, e_2, e_3, e_4\}$

Table 10
Calculation of $\textbf{almost\_all}_E((tall)_{\alpha_1}, (blond)_{\alpha_2})$

$\textbf{almost\_all}_E((tall)_{\geqslant \alpha_1}, (blond)_{\geqslant \alpha_2})$	$\alpha_2 \in (0.8, 1]$	$\alpha_2 \in (0.3, 0.8]$	$\alpha_2 \in (0.1, 0.3]$	$\alpha_1 \in (0, 0.1]$
$\alpha_1 \in (0.9, 1]$	0.02:0	0.05:0	0.02:1	0.01:1
$\alpha_1 \in (0.8, 0.9]$	0.02:0	0.05:0	0.02:1	0.01:1
$\alpha_1 \in (0.2, 0.8]$	0.12:0	0.3:0.33	0.12:1	0.06:1
$\alpha_1 \in (0, 0.2]$	0.04:0	0.1:0	0.04:0.5	0.02:1

This mechanism can be interpreted [10] from the perspective of the *mass assignment theory* [1] or *random sets theories* [22].

Example 24. The same example is considered as for the mechanism M. In order to evaluate the sentence we first evaluate the α-cuts of *tall* and *blond*, shown in Table 9. Table 10 shows the application of the semi-fuzzy quantifier $\textbf{almost\_all}_E$ to the different combinations of α-cuts. Then, the result of evaluating the sentence being

$$\begin{aligned} F^I(\textbf{almost\_all}_E)(X_1, X_2) &= 0.02 \times 0 + 0.05 \times 0 + 1 \times 0.02 + \cdots + 1 \times 0.02 \\ &= 0.379 \end{aligned}$$

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